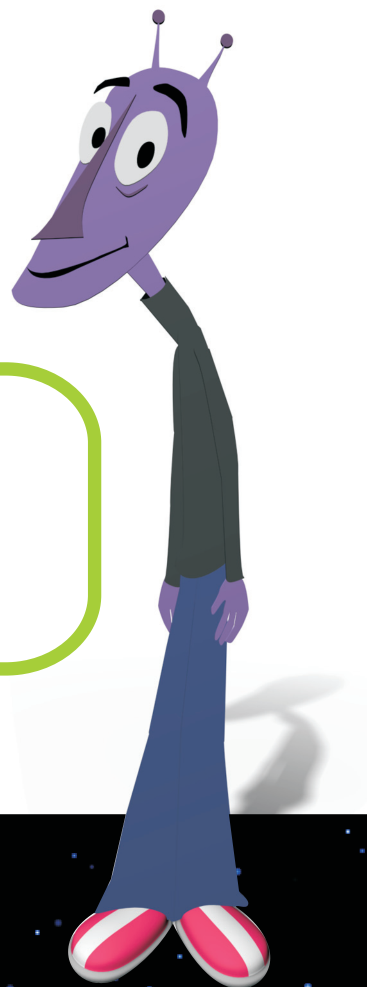
Numbers and Operations

Module 7

Ratio, Proportion, and Percent



Lesson 4
Ratios, Rates, and Proportional Reasoning

Teacher Notes 7.4

Objectives

- Write ratios as fractions in simplest form.
- Determine unit rates.
- Use proportional reasoning and ratios to represent problem situations.

Vocabulary

Ratio (4.1)
Simplest form (4.5)
Rate
Unit rate
Unit cost
Proportion (7.2)
Means (7.2)
Extremes (7.2)
Cross products (7.2)

Prerequisites

Writing ratios and proportions

Multiplying and dividing whole numbers and decimals

Simplifying fractions

Get Started

- Write the following pattern on the board and have students fill in the missing numbers. Ask students how they found their answers.

  4, 8, 12, ____, 20, 24, ____

  16, 28: Keep adding four.

- Write the following table on the board and have students fill in the missing numbers. Ask students how they found their answers.

2	4	6	8	**10**	12	14
5	10	**15**	20	25	30	**35**

In the first row, the pattern is adding two. In the second row, the pattern is adding five.

- Write the following table on the board and have students fill in the missing numbers. Ask students how they found their answers.

3	7
6	14
9	21
12	**28**
15	35
18	42
21	49

In the first column, the pattern is adding three. In the second column, the pattern is adding seven.

- Tell students they can use patterns when reasoning proportionally and will do so in this lesson.

Subtopic 1 — Ratios, Rates, and Unit Rates

Expand Their Horizons

In this subtopic, students review rates and ratios before exploring unit rates and unit costs. Like a fraction, a ratio is in simplest form when the terms have no common factors other than one. A rate is a ratio in which the units in the numerator are different than the units in the denominator. Often, the unit in the second quantity is a unit of time although this is not a necessary condition for a ratio to be a rate. If the second quantity is one, the rate is a unit rate.

As shown in the lesson, when asked to write a rate or unit rate, there are two possibilities although one is probably used more regularly than the other. For example, if it takes Ronnie three minutes to pump 12 gallons of gas, the rate can be either $\frac{1 \text{ min}}{4 \text{ gal}}$ or $\frac{4 \text{ gal}}{1 \text{ min}}$, read as one minute per four gallons or four gallons per minute. More likely someone will want to know the rate as the number of gallons per minute rather than the number of minutes per gallon. Sometimes the directions will specify which to use by asking; for instance, find the unit rate per minute.

One type of unit rate is unit cost. It gives the price of one item. For example, if three melons cost \$6, set up the rate $\frac{\$6}{3 \text{ melons}}$ and divide both the numerator and denominator by three to make the rate a unit rate: $\frac{\$2}{1 \text{ melon}}$. One melon costs \$2.

Common error alert:

Students often mix up the numerator and denominator when finding unit rates. For example, in the previous example, they may write $\frac{3\text{ melons}}{\$6}$ because that is the order in which the situation was phrased. Remind them that they are looking for the cost *per* item; where cost is the money, *per* means divide, and *item* is the number of items.

1 Write the number of roses as the first term and the number of tulips as the second term. Divide out the common factor of four.

2 This problem asks for a unit cost. Write the rate as the cost divided by the number of sodas. To make the denominator one, divide the numerator and denominator by 12. The new numerator is the cost per soda.

Additional Examples

1. Rob nailed up 20 boards in eight minutes. Find the unit rate per minute.

Per minute means to divide by the minutes.

$$\frac{\text{boards}}{\text{minutes}} = \frac{20}{8}$$

Divide both terms by eight.

$$\frac{20 \div 8}{8 \div 8} = \frac{2.5}{1}$$

The unit rate is 2.5 boards per minute.

2. Rosemary counted 26 ducks and 18 squirrels at the park. Write the ratio of squirrels to ducks as a fraction in simplest form.

The question asks for the ratio of squirrels to ducks, so write the ratio in the form $\frac{\text{squirrels}}{\text{ducks}}$ and divide out the common factor of two.

$$\frac{18}{26} = \frac{9}{13}$$

The ratio of squirrels to ducks is $\frac{9}{13}$.

Expand Their Horizons

In this subtopic, students solve real-life problems by reasoning proportionally. Writing a proportion and finding the missing term is a method that will work for all the problems in this subtopic. However, for some problems, students may prefer either to find the unit rate and then to multiply or to use the “factor of change” method. One problem and three methods of solving it are shown below.

Problem: Sally burned 320 calories in 40 minutes. At this rate, how many calories will she burn in 120 minutes?

Write and Solve a Proportion: $\frac{\text{calories}}{\text{minutes}} \begin{matrix}\longrightarrow \\ \longrightarrow\end{matrix} \frac{320}{40} = \frac{?}{120}$

$$320 \times 120 = 40 \times ?$$
$$38,400 = 40 \times ?$$
$$960 = ?$$

Find the Unit Rate and Multiply: Find the number of calories burned in one minute.

$$\frac{\text{calories}}{\text{minutes}} \begin{matrix}\longrightarrow \\ \longrightarrow\end{matrix} \frac{320 \div 40}{40 \div 40} = \frac{8}{1}$$

Sally burned eight calories in one minute. Multiply eight by 120: 8 × 120 = 960.

Use the “factor of change” method: Think: The factor of change between 40 minutes and 120 minutes is three. Use the same factor of change for the number of calories: 320 × 3 = 960. This method can also be called the “times as many” method. Therefore, *times as many,* between the minutes, is the same as between the calories. That is, 120 minutes is three times as many as 40 minutes and 960 calories is three times as many as 320 calories.

Teachers may wish to present this problem and the three solutions to the class and may ask which method(s) are most appropriate for this problem and why. While all lead to a correct answer of 960 calories, the last two methods are quicker and use smaller numbers than the first method.

When solving by writing a proportion, students can write the terms in either order, as long as it is the same in both ratios. For instance, in Sally’s situation, we could have used $\frac{\text{minutes}}{\text{calories}} \rightarrow \frac{40}{320} = \frac{120}{?}$. Notice that the cross products will be the same as before.

Common Error Alert:

After writing the first ratio in a proportion, students might switch the order of the terms in the second ratio. Both ratios need to be written with the same units in the same place. Tell them that writing the ratio in words off to the side can help remind them of which terms belongs where.

3 Write a proportion using peanuts and coconuts in the same order in both ratios. Because 12 ÷ 4 = 3, students should find the unknown by dividing five by four.

4 Write a proportion using beats and seconds in the same order in both ratios. The numbers are not compatible, so find the product of the means and extremes and divide to find the unknown. An alternate method is to factor four out of the first ratio. Then it becomes very easy to multiply both the numerator and denominator by 10.

If solving by finding a unit rate, the number of beats per second is $1\frac{2}{3}$. Multiply this by 60.

$$1\frac{2}{3} \times 60$$

$$\frac{5}{\cancel{3}_1} \times \frac{\cancel{60}^{20}}{1} = \frac{100}{1} = 100$$

Additional Examples

1. It cost Bobby $4.50 to mail a two-pound package to his brother at camp. At this rate, how much would it cost him to mail a package that weighs eight pounds?

Using the "factor of change" method: Eight pounds is four times as many as two pounds. Multiply $4.50 by four.

$$\$4.50 \times 4 = \$18.00$$

Finding the unit rate:

$$\frac{\text{cost}}{\text{pounds}} : \frac{\$4.50 \div 2}{2 \div 2} = \frac{\$2.25}{1}$$

Multiply the unit cost by eight.

$$\$2.25 \times 8 = \$18.00$$

2. Ms. Jimenez drove 350 miles on 14 gallons of gas. At this rate, how many miles can she drive with eight gallons of gas?

Writing and Solving a Proportion:

$$\frac{\text{miles}}{\text{gallon}} : \frac{350}{14} = \frac{?}{8}$$

$$350 \times 8 = 14 \times ?$$

$$2{,}800 = 14 \times ?$$

$$2{,}800 \div 14 = 200$$

Finding the unit rate:

$$\frac{\text{miles}}{\text{gallon}} : \frac{350 \div 14}{14 \div 14} = \frac{25}{1}$$

Multiply the unit rate by eight.

$$25 \times 8 = 200$$

Expand Their Horizons

In this subtopic, students are shown how to use tables to compare rates. To make a table, form a pattern with the rates. For example, if Ellen makes three batches of cookies in two hours, then she makes six batches of cookies in four hours and nine batches of cookies in six hours; the information in table form would look like this.

batches	hours
3	2
6	4
9	6

The number of batches increases by three in each row while the number of hours increases by two.

Now suppose Shirley makes four batches of cookies in three hours. Her table is shown below.

Ellen

batches	hours
3	2
6	4
9	6

Shirley

batches	hours
4	3
8	6
12	9

Study each table to find if the number of batches or the number of hours is ever the same. Both tables have six hours. In six hours, Ellen made nine batches of cookies while Shirley made eight. Ellen made more than Shirley in the same amount of time; she is making cookies at a faster rate.

The number of rows needed per table will vary with each problem. Tell students they need to extend their tables until they see the same number in either of the two categories.

Common Error Alert:

Students may answer incorrectly by not looking at the title of each column in the table. The time can be placed in either column. Tell students to always double check what each column stands for before interpreting the results.

For each person, the unit rate can be found by dividing the number of batches by the time. For example, Ellen's rate is three batches for every two hours, or $1\frac{1}{2}$ batches per hour. The unit rate is the same for every row in a table. For Ellen: $\frac{3}{2}=1\frac{1}{2}$, $\frac{6}{4}=1\frac{1}{2}$, $\frac{9}{6}=1\frac{1}{2}$. Shirley's unit rate is $\frac{4}{3}=1\frac{1}{3}$. Notice that Ellen's unit rate is greater than Shirley's unit rate. This confirms the previous conclusion that Ellen made cookies at a faster rate.

Students may notice that making rows until the same number appears in both is like finding a least common multiple. This is no coincidence. Rates are ratios and ratios are fractions. Finding the least common multiple is the same as finding the least common denominator, and then the numerators are compared. For our cookie problem above, Ellen making three batches in two hours is $\frac{3}{2}$, and Shirley making four batches in three hours is $\frac{4}{3}$. Write each fraction with the least common denominator: $\frac{3}{2}=\frac{9}{6}$ and $\frac{4}{3}=\frac{8}{6}$. Similar to before, we see that nine is greater than eight, so the first fraction (or rate) is greater.

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In Gina's table, the numbers of tablespoons keep increasing by three, and the ounces of milk keep increasing by four. In Rita's table, the numbers of tablespoons increase by four and the ounces of milk by five. Twelve tablespoons occur in both tables. Interpret the meaning. The same amount of syrup in 15 ounces of milk will be stronger than in 16 ounces of milk.

Point out that if Gina's table were to be extended another row, the ounces of milk would be 20 which can be compared with 20 ounces of milk in Rita's table. For the same amount of milk, more syrup means a stronger chocolate taste. In 20 ounces of milk, Gina would use 15 tablespoons of syrup while Rita would use 16. Rita still makes a stronger chocolate drink.

Additional Examples

1. **Peter delivered 20 newspapers in 30 minutes. Faith delivered 30 newspapers in 45 minutes. Use a table to determine who makes their deliveries faster.**

2. **Taxi Plus charges \$4 for every $\frac{1}{2}$ mile traveled. Going Places Taxi charges \$2.50 for every $\frac{1}{4}$ mile traveled. Which taxi company is more expensive?**

continued on next page

Peter

papers	min
20	30
40	60
60	90
80	120

Faith

papers	min
30	45
60	90
90	135
120	180

Both deliver papers at a rate of 60 papers in 90 minutes. They make their deliveries at the same rate.

Taxi Plus

cost	miles
$4	$\frac{1}{2}$
$8	1
$12	$1\frac{1}{2}$
$16	2

Going Places Taxi

cost	miles
$2.50	$\frac{1}{4}$
$5.00	$\frac{1}{2}$
$7.50	$\frac{3}{4}$
$10.00	1

For one mile, Taxi Plus charges $8 and Going Places charges $10. Going Places is more expensive.

Notice that these are the unit rates:

$$\frac{\$4 \times 2}{\frac{1}{2} \times 2} = \frac{\$8}{1} \text{ and } \frac{\$2.50 \times 4}{\frac{1}{4} \times 4} = \frac{\$10}{1}.$$

Look Beyond

The tables shown in the last subtopic are similar to tables which represent linear functions in algebra. As *x* increases by a constant amount, *y* increases by a constant amount. In the example below, as *x* increases by three, *y* increases by five.

x	3	6	9	12
y	5	10	15	20

In algebra, students will use tables such as these to determine if a function is linear, quadratic, or cubic. If it is linear, they can also use the table to find the slope of the line by choosing two rows, by subtracting the *y*-values, by subtracting the *x*-values, and then by writing the ratio of the differences. For the example above, the slope is $\frac{10-5}{6-3} = \frac{5}{3}$.

Connections

Proportional reasoning is a necessary skill in many areas of our lives. For instance, if the scale on a map states that one-half inch equals three miles, drivers can write and can solve a proportion to find how many miles separate two towns that are $5\frac{1}{4}$ inches apart.

If a recipe calls for three teaspoons of salt for every two teaspoons of sugar, chefs can use the factor of change method to find how many teaspoons of salt to use if they are using six teaspoons of sugar. Lastly, if an item is sold both in units of six and 10, consumers can find the unit cost of each item to determine which package is the better buy.