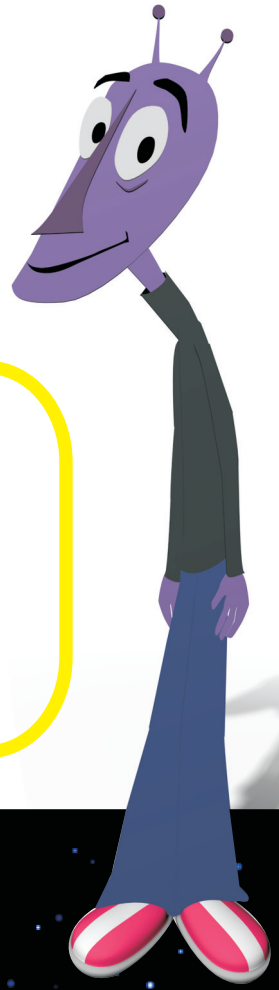
Measurement

★ Module 13 ★

Perimeter, Area, and Volume



Lesson 7
Volume:
Pyramids and Cones

Teacher Notes 13.7

Objectives

- Derive and use formulas for volume of pyramids and cones and justify using geometric models and common materials.
- Use cubic units to find the volume of pyramids and cones.
- Demonstrate understanding of when to use linear units to describe perimeter, square units to describe area or surface units, and cubic units to describe volume, in real-world situations.
- Compare and contrast the differences among linear units, square units, and cubic units.

Vocabulary

Volume (12.1)
Cone (10.4)
Three-dimensional (10.4)
Circle (9.3)
Radius (9.3)
Cylinder (10.4)
Order of Operations (1.1)
Pi (9.3)
Diameter (9.3)
Pyramid (10.4)
Prism (10.4)
Area (7.1)
Right triangle (8.4)
Base (10.4)

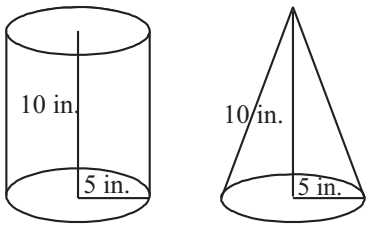
Prerequisites

Performing operations with whole numbers, decimals, and fractions

Evaluating expressions using the order of operations

Get Started

- On the board, draw a cylinder and a cone so that both have a height of 10 inches and a radius of five inches.



- Have students find the volume of the cylinder. **About 785 in.^{3}**

- Ask students if they think the volume of the cone is greater than or less than the volume of the cylinder and why.
  Possible answer: less than because the cone could fit inside the cylinder

- Have students estimate the volume of the cone. Write students' guesses on the board. Find out whose estimates are most accurate.

- Tell students that during the lesson they will learn how the volumes of the two figures are related and then find out whose estimates are most accurate.

Subtopic 1 — Volume of a Cone

Expand Their Horizons

In this subtopic, students find the volume of cones. They learn, for a cone and a cylinder of the same height and radius, the volume of the cylinder is three times the volume of the cone. In other words, the volume of the cone is one-third the volume of the cylinder. Because the volume of a cylinder can be found by multiplying the area of the base by the height, the volume of a cone can be found by taking one-third of the product of the area of the base and height: $V = \frac{1}{3}\pi r^2 h$. Because multiplying by one-third is the same as dividing by three, the formula can also be written as $V = \frac{\pi r^2 h}{3}$.

After viewing the first subtopic, have students find the volume of the cone in the Get Started activity. The volume is about 261.7 cubic inches. Check the original estimates to see which were the most accurate. Have those students tell how they had determined their estimates.

Common Error Alert:

When using the formula for the volume of a cone, students may divide out factors in the numerator and denominator that are not allowed to be divided out. For example, if the radius is a multiple of three, students may divide the three in the denominator with the radius rather than the radius squared.

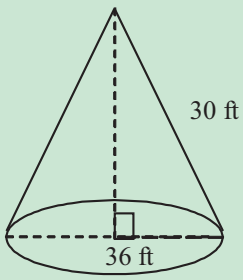
Incorrect Simplifying	Correct Simplifying
$V = \frac{1}{3}\pi r^2 h$	$V = \frac{1}{3}\pi r^2 h$
$V = \frac{1}{\cancel{3}_1}\pi\left(\cancel{6}^2\right)^2 5$	$V = \frac{1}{3}\pi (6)^2 5$
$V = \pi(4)^2 5$	$V = \frac{1}{\cancel{3}_1}\pi\left(\cancel{36}^{12}\right) 5$
$V = 80\pi$	$V = 60\pi$

1 In the formula, substitute three for the radius and 11 for the height: $V = \frac{1}{3}\pi(3)^2(11)$. Square the radius: $\frac{1}{3}\pi(9)(11)$ and simplify: $33\pi \approx 103.62$. The volume of the cone is about 103.62 cubic inches.

2 Find the volume of the can using the formula $V = \pi r^2 h$. The volume is 275π cubic centimeters. Find the volume of the bag using the formula $V = \frac{1}{3}\pi r^2 h$. The volume is 54π cubic centimeters. Divide the volumes: $\frac{275\pi}{54\pi}$. First divide out π: $\frac{275}{54}$. Then, divide the numbers: $275 \div 54 \approx 5.09$. The can will fill the bag five times.

Additional Examples

1. What is the volume of the cone?



The slant height is given. First, find the height by using the Pythagorean Theorem. The radius is $36 \div 2 = 18$ feet.

$$18^2 + h^2 = 30^2$$
$$324 + h^2 = 900$$
$$h^2 = 576$$
$$h = 24$$

Then, use the formula.

$$\begin{aligned} V &= \tfrac{1}{3}\pi r^2 h \\ &= \tfrac{1}{3}\pi (18)^2(24) \\ &= \tfrac{1}{3}\pi (324)(24) \\ &= 2{,}592\pi \\ &\approx 8{,}138.88 \end{aligned}$$

The volume of the cone is about 8,138.88 cubic feet.

2. The volume of a cylinder is 2,369 cm^3. What is the volume of a cone with the same height and radius as the cylinder?

For a cone and cylinder of the same height and radius, the volume of the cone is one-third the volume of the cylinder.

$$2{,}369 \div 3 \approx 789.67$$

The volume of the cone is about 789.67 cubic centimeters.

Expand Their Horizons

In this subtopic, students learn the formula for the volume of a pyramid. Analogous to the relationship between a cylinder and cone, when a pyramid and a prism have the same base area and height, the volume of the pyramid is one-third the volume of the prism. Because the volume of a prism is found by multiplying the area of the base times the height, the area of a pyramid can be found by taking one-third the product of the area of the base and the height of the pyramid.

3 Multiply the area of the base, 24, by the height, five: $24 \times 5 = 120$. Divide by three: $120 \div 3 = 40$. The volume is 40 cubic meters.

4 Find the area of the triangular base: $\frac{1}{2}(6)(8) = 24$. Multiply by the height: $24 \times 9 = 216$. Divide by three: $216 \div 3 = 72$. The volume is 72 cubic meters.

Additional Examples

1. A square pyramid has a height of six inches and a volume of 18 cubic inches. Find the perimeter of the base.

Substitute what is known into the formula and solve for the area of the base.

$V = \frac{1}{3}Bh$

$18 = \frac{1}{3}B(6)$

$18 = B(2)$

$9 = B$

The base has an area of nine square inches. Because the base is a square, the length of one side of the base is the square root of nine: $\sqrt{9} = 3$.

The perimeter of a square with a side length of three inches is $4(3) = 12$ inches.

2. A pyramid has a height of nine meters and a base in the shape of a right triangle. The legs of the right triangle measure five meters and eight meters. Find the volume of the pyramid.

Find the area of the triangular base: $A = \frac{1}{2}(5)(8) = 20\ m^2$.

Use the formula for the volume of a pyramid, substitute 20 for B and 9 for h.

$V = \frac{1}{3}Bh$

$= \frac{1}{3}(20)(9)$

$= \frac{1}{3}(180)$

$= 60$

The volume is 60 cubic meters.

Look Beyond

In high school geometry, students will be able to find the volume of triangular pyramids whose bases are not right triangles by using trigonometry to find the heights of the triangular bases. They will also be able to find the volume of a pyramid whose base is any regular polygon. For example, they may find the volume of a regular hexagonal pyramid.

Connections

In the parts of the country where it usually snows in the winter, road salt is stored in large quantities in structures near major highways. The intrinsic properties of de-icing salt allow it to naturally form a conical shape when it is poured. Highway crew managers must be able to accurately estimate the volume of salt in a salt pile to know if they have enough salt to de-ice the roads during an upcoming storm.