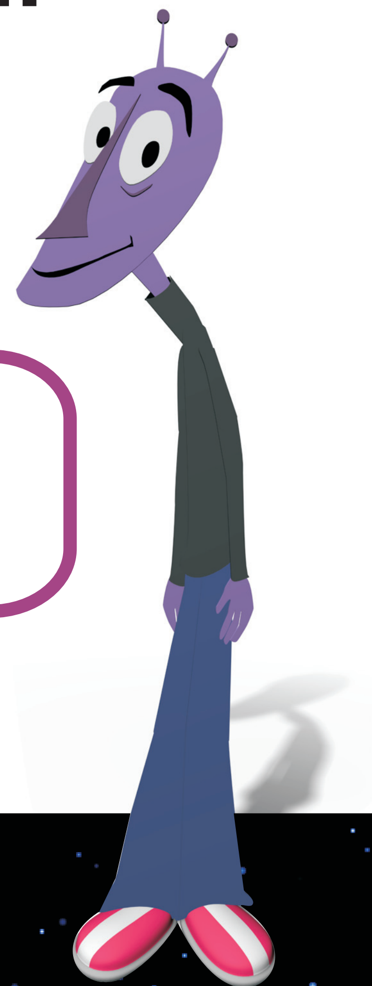
Geometry

Module 10

Coordinate Geometry and Spatial Visualization



Lesson 3
Coordinate Geometry

Teacher Notes 10.3

Objectives

- Use coordinate geometry to explore the links between geometric and algebraic representations of problems (lengths of segments/distance between points, slope/perpendicular-parallel lines).
- Count the distance between two points on a horizontal or vertical line and compare the lengths of the paths on a grid.
- Find the distance between two points on a number line.
- Find the distance between two points on a number line and locate the midpoint.
- Find the distance between two points on a coordinate plane using the Pythagorean Theorem.

Vocabulary

Coordinate plane (10.1)
Absolute value (3.1)
Midpoint
Equidistant (9.3)
Right angle (8.4)
Pythagorean Theorem (8.7)
Hypotenuse (8.7)
Slope
Rise
Run
Parallel lines (8.2)
Perpendicular lines (8.2)
Opposite numbers (3.1)
Reciprocal (6.6)

Prerequisites

Plotting points on a coordinate plane

Adding and subtracting integers

Finding the absolute value of a number

Using the Pythagorean Theorem

Get Started

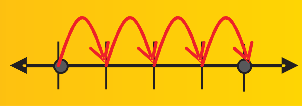
- Write the following numbers on the board:
  3, 4, 5, 6, 8, 9, 10, 12, 13, 15, 16, 20.
- Ask students how they determine whether a triangle is a right triangle. **The sum of the squares of the legs equals the square of the hypotenuse.**
- Have students work alone, or in pairs, to find which combinations of numbers above can be used as the sides of a right triangle.
  3, 4, and 5
  6, 8, and 10
  5, 12, and 13
  9, 12, and 15
  12, 16, and 20

Expand Their Horizons

In this subtopic, students first find distances on a number line. The distance between two coordinates on a number line is the absolute value of the difference between the coordinates. Absolute value is used because it does not make sense for a length to be negative. Explain to students that if they do not move, they have moved zero units. This is the least movement they can make. Moving backwards is still moving; it just happens to be in a different direction than forward.

Common Error Alert:

Students whose distances are always one less than the correct answer may be counting the number of hash marks between coordinates instead of the number of spaces between coordinates. Show students how to count spaces by drawing arcs from one hash mark to the next. In the figure below, the distance between the two points is four.



Students also learn about the midpoint of a segment. As its name implies, a midpoint is the "middle point" of a segment. It divides the segment into two congruent segments. The coordinate of a midpoint can be found by finding the average of the coordinates. In other words, add the coordinates and divide by the number of coordinates (two).

Finding distances is moved from one-dimensional space to two when students are asked to find the distance between two points on a coordinate plane. If the points are on a horizontal or vertical line, students can either count units or subtract either the *x*- or *y*-coordinates. Otherwise, they must treat the segment that connects the points as the hypotenuse of a right triangle and use the Pythagorean Theorem.

When using the Pythagorean Theorem, have students write all their answers as exact answers. That means, for distances that are not whole numbers, write the answer as a square root. For example, if $c^2 = 17$, then $c = \sqrt{17}$ and the distance is $\sqrt{17}$ units.

1 There are seven spaces between *C* and *D*. If subtracting rather than counting, $1 - (-6) = 1 + 6 = 7$. The distance is seven units. To find the coordinate of the midpoint, add the coordinates and divide by two. The coordinate of the midpoint is -2.5, or $-2\frac{1}{2}$. This point is $3\frac{1}{2}$ units from *C* and $3\frac{1}{2}$ units from *D*.

2 Connect the points. Then, draw two segments that are legs of a right triangle, with this segment as its hypotenuse. There are two possibilities: connect (-4, -6) to (1, -6) and (1, -6) to (1, 6), as shown in the lesson, or connect (-4, -6) to (-4, 6) and (-4, 6) to (1, 6). Either way, the two legs will have lengths of five and 12 units. Use the Pythagorean Theorem to find that the length of the hypotenuse is 13 units.

Additional Examples

1. Find the coordinate of the midpoint of $\overline{GH}$ if the coordinate of *G* is -12 and the coordinate of *H* is -3.

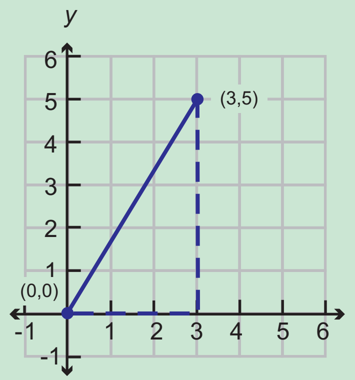
Add the coordinates of the endpoints and divide by two.

$$\frac{-12+(-3)}{2}=\frac{-15}{2}=-7\frac{1}{2}$$

The coordinate of the midpoint is $-7\frac{1}{2}$.

2. Find the distance between the origin and (3, 5).

Draw a right triangle so that the hypotenuse extends from (0, 0) to (3, 5).



The lengths of the legs are three units and five units. Use the Pythagorean Theorem to find the hypotenuse.

$$3^2+5^2=c^2$$
$$9+25=c^2$$
$$34=c^2$$
$$\sqrt{34}=c$$

The distance from the origin to (3, 5) is $\sqrt{34}$ units.

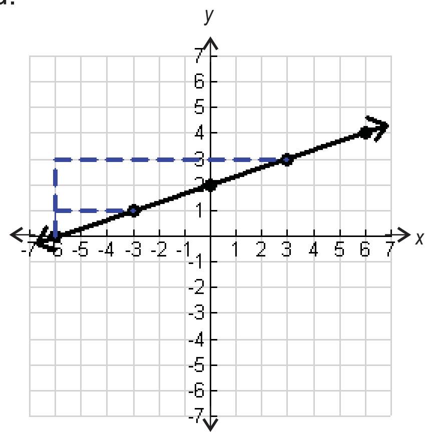
Subtopic 2 Slope

Expand Their Horizons

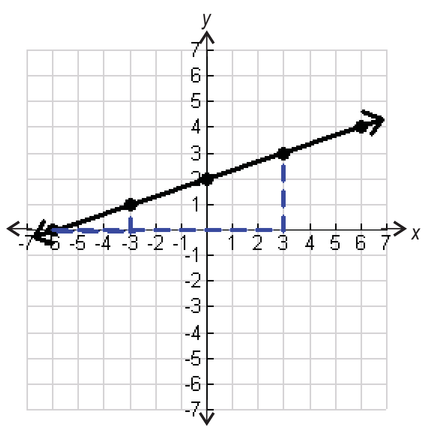
In this subtopic, students find slopes of lines. Slope is defined as the steepness of a line. On a coordinate plane, it is found by choosing two points on the line and by forming a ratio of the difference of the *y*-coordinates to the difference of the *x*-coordinates. More simply put, it is $\frac{\text{change in } y}{\text{change in } x}$, or $\frac{\text{rise}}{\text{run}}$.

Explain that the slope of a line is the same regardless of which two points are chosen. To keep calculations simple and accurate, always choose points whose coordinates are whole numbers.

One way to find the slope of the line below is to find the ratio of rise to run between (-6, 0) and (-3, 1). Similar to finding the distance between two points, students can draw legs of a right triangle so that part of the line forms the hypotenuse. To get from (-6, 0) to (-3, 1), follow the dotted line segments to rise one unit and run three units making the slope $\frac{1}{3}$. Alternatively, they can use the points (-6, 0) and (3, 3). In this case, the rise is three, and the run is nine, making the slope $\frac{3}{9}$, which simplifies to $\frac{1}{3}$. Have students use this picture to find the slope between other pairs of points to see that the rise to run ratio always simplifies to one-third.



There are two ways to form the legs of the right triangle, as shown in the diagram below. Rise and run can both be either positive or negative. To find the slope using (3, 3) and (-6, 0), the rise is -3 (moving down), and the run is -9 (moving left). The slope ratio, also represented by *m*, still simplifies to $\frac{1}{3}$.



Show students the graph of a horizontal line and label the coordinates. The *y*-coordinates are the same, making the change in *y*, which is the rise, zero. A fraction with a numerator of zero equals zero. Then, show students the graph of a vertical line and label the coordinates. The *x*-coordinates are the same, making the change in *x*, which is the run, zero. A fraction with a denominator of zero is undefined.

Graph a line with a slope of two and a slope of four. The line with a slope of four is steeper because it rises four units for every run of one unit, whereas the other line rises only two units for the same amount of run. Likewise, a line with a slope of $\frac{1}{2}$ is steeper than a line with a slope of $\frac{1}{4}$. The former slope runs only two units for every one unit of rise, while the latter runs four units for the same amount of rise. In brief, the line with the greater absolute value of slope is the steeper of the two lines. It is important to mention absolute value because a line with a slope of $\frac{1}{2}$ has the same steepness as one with a slope of $-\frac{1}{2}$. Though they slant in different directions, the amount of slant is the same.

Lines that slant upward from left to right always have a positive slope. Show students a line with a positive slope on a coordinate plane. Show how as the *x*-coordinates increase (move from left to right on the *x*-axis), the *y*-coordinates also increase (move up on the *y*-axis). This can also be said in reverse: as the *x*-coordinates decrease (move from right to left on the *x*-axis), the *y*-coordinates also decrease (move down on the number line). Regardless of which direction students move on the *x*-axis, the *y*-coordinates change the same way. That is, both *x* and *y* increase, or both *x* and *y* decrease.

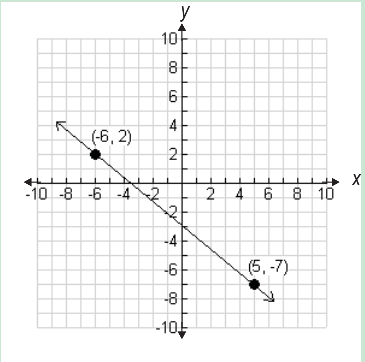
Lines that slant downward from left to right have a negative slope. Unlike what happened above for positive slope, when a line has a negative slope, *x* and *y* change in opposite directions. As *x* increases, *y* decreases, and as *x* decreases, *y* increases.

3 Form the ratio of rise to run. The rise is -8 and the run is two. The ratio is $\frac{-8}{2}$, which simplifies to -4. Segments can also be drawn on the right side, making a rise of eight with a run of -2. Then the ratio is $\frac{8}{-2}$, which still simplifies to -4.

4 Use the ratio $\frac{5}{4}$ or $\frac{-5}{-4}$. The slope is $\frac{5}{4}$. Do not rewrite this as a mixed number.

Additional Examples

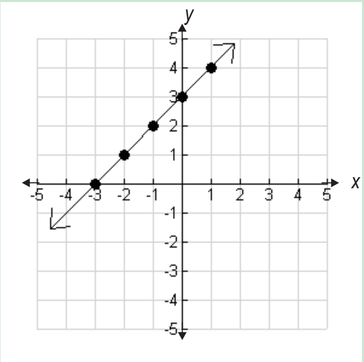
1. Find the slope of the line.



Starting from (-6, 2) and moving to (5, -7), the rise is -9 and the run is 11.

The slope is $\frac{-9}{11}$, or $-\frac{9}{11}$.

2. Find the slope of the line.



Choose any two points. To move from (-2, 1) to (-1, 2), rise one and run one.

The ratio of any two points on the line is, or simplifies to, one. The slope of the line is one.

Expand Their Horizons

In this subtopic, students learn that the slopes of parallel lines are equal. Draw a large coordinate plane on the board and have a few volunteers come up. Have each student start at any point on the plane and draw a dot there. Then, tell each of them to simultaneously rise up two and run over three from that point and draw a dot. Repeat two or three times. Have each student connect their points. All of the lines will be parallel.

Slopes of perpendicular lines are opposite reciprocals. That is, they have a product of -1. Draw a pair of perpendicular lines on a coordinate plane. Remind students that for two numbers to have a product of negative one, one factor must be positive and the other must be negative. Then, point out that the slope of one of the lines is positive and that the other is negative because one slants upward from left to right while the other slants downward from left to right. Also, show on the graph how the absolute values of the rise and run are switched. The absolute value of the rise for one of the lines is the absolute value of the run for the other line, and vice versa.

5 The slope of line *t* is $\frac{3}{5}$. Any line parallel to line *t* has the same slope. To find the slope of any line perpendicular to line *t*, switch the rise and run and use the opposite sign: $-\frac{5}{3}$.

Additional Examples

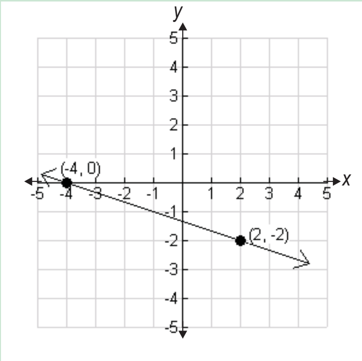
1. **Find the slope of any line perpendicular to a line with a slope of -1.**

The reciprocal of one is one. The opposite of -1 is one.

Another way to find the slope is find what number times -1 equals -1. The answer is one.

The slope of any line perpendicular to a line with a slope of -1 is 1.

2. **Find the slope of any line parallel to the line shown below.**



The slope of the line is $\frac{-2}{6}$, which simplifies to $-\frac{1}{3}$.

The slope of any line parallel to the line will be the same: $-\frac{1}{3}$.

Look Beyond

Students will use slope in algebra when they graph lines in the coordinate plane. The equation of a line can be written as $y = mx + b$, where m is the slope. Students will use the relationship between the slopes of parallel and perpendicular lines in high school geometry when they learn how to write a coordinate proof. They will use the concept of "as x increases, y increases" and "as x increases, y decreases" in statistics classes when they determine if there is a positive or negative relationship in a scatter plot.

Connections

The slope of a roof is called *roof slope*. A roof should have a slope so that water from rain and snow can run off. If the water cannot run off, and it does not evaporate quickly, the water can penetrate the roof, which can slowly rot the underlying wood and cause it to collapse.

Roof slope is always given in inches where the run is 12. A roof slope of $\frac{3}{12}$ or less is considered low slope. Roofs with a slope of $\frac{8}{12}$ or more can be difficult to walk on. When constructing a roof, building inspectors use roof slopes to determine if the correct safety equipment, such as harnesses, should be used.