20.4 teacher notes

Objectives

- Determine whether a graph is traversable.
- Find a traversable path.
- Determine whether two graphs are equivalent.

Vocabulary

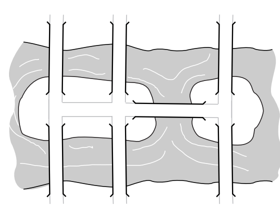
Graph
Vertex
Edge
Degree of a vertex
Path
Traversable graph
Equivalent graphs

Prerequisite

Determining whether a number is even or odd

Get Started

- Separate the class into pairs or groups of three to solve a mathematical puzzle.
- Draw the image below on the chalkboard.



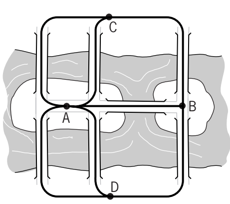
- Say, “There was a city in Europe that had seven bridges connecting two islands to the banks of a river. It was a popular pastime for the residents to try to take a walk and cross all seven bridges without crossing any bridge more than once. Can you find a way?”
- Give groups a few minutes to try to find a solution to the problem.
- Tell the class that this is the famous Konigsberg Bridge Problem, and it has been proven that it has no solution. Also, share with the class that this problem provided the basis for the branch of mathematics called graph theory, which they will study in today’s lesson.

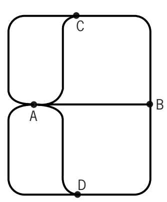
Section 1

Expand Their Horizons

Discrete Mathematics is the branch of mathematics that deals with decision making and objects that can be counted. In Section 1, students will learn key concepts in the field of graph theory. Students will learn a graph is a collection of points, called vertices, connected by line segments or arcs, called edges. They will learn to identify the degree of a vertex on a graph as well as determine whether a graph is traversable.

After the terms "vertex" and "edge" are defined on screen, pause to ask the class to compare and contrast the use of the words "graph," "vertex," and "edge" in this lesson and in previous lessons. Point out that the meanings of these terms are slightly different in graph theory than in algebra and geometry. Show students how to redraw the image in the opening activity as a graph.





Students should now identify the vertices and edges. (The vertices are A, B, C, and D; the vertices correspond to land masses. The edges are the paths connecting the vertices.) Ask students to find the degree of each vertex. (A has degree 5, B has degree 3, C has degree 3, and D has degree 3.)

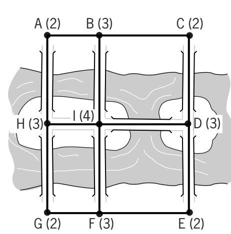
Some students will find it helpful to create their own graphs based on a given description. Ask the students to create a graph with three vertices and four edges and then, have volunteers draw their graphs on the chalkboard.

Finding traversable paths is one of the most useful applications in graph theory. Ask students to think of other situations in which the identification of a traversable path would make a job easier. Students may give a newspaper or delivery route as an example.

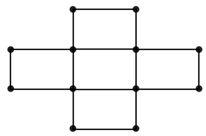
Common Error Alert

Students may have trouble finding traversable paths if they forget which edges have already been traced. Encourage students to develop methods of keeping track of edges, such as drawing an "x" over edges that have already been traced.

Students should explain why the Konigsberg Bridge Problem mentioned in the opening activity has no solution. (A path is traversable if and only if one of the following conditions is met: Every vertex has an even degree, or exactly two vertices have an odd degree. In the Konigsberg Bridge Problem, all four vertices have an odd degree.) Note that this graph is not unigue, but no similar graph of the Konigsberg Bridge Problem will have a traversable path either. Encourage students to make a different but similar map of the Konigsberg Bridge Problem and verify that it has no traversable path. Here's a similar graph with the degree beside each vertex. Notice there are four vertices with odd degrees so this graph is not traversable.



Use the following graph for Questions 1 and 2. The graph shown represents a neighborhood. The edges represent the streets, and the vertices represent the intersections.



1 **Find the degree of each vertex.** Encourage students to write the degree of each vertex in the graph next to that vertex. Students will be asked in the next Guided Notes problem if there is a traversable path. Having the degree next to each vertex will be helpful.

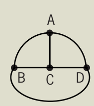
Students may enjoy creating graphs of their own neighborhoods. Students could create graphs similar to the one used in Guided Notes Problem 1. Students can identify the degree of each vertex in their own graphs.

2 **A student in the neighborhood is selling cookies from door to door. Is there a traversable path that would enable her to walk around the entire neighborhood without walking any part of a street more than once?** This graph is traversable because every vertex has an even degree. As an additional exercise, ask the students to find a traversable path in the graph, assigning each student a different starting point. Remember that when all the vertices have even degrees, any vertex can be the starting point of a traversable path.

Ask students if it is possible to make the graph used in Guided Notes Problems 1 and 2 not traversable by adding one edge and keeping the same vertices. Give them a few minutes to think about it. (It is not possible because adding one edge and keeping the same vertices will give the graph exactly two vertices of odd degree, and the graph will still be traversable.)

Additional Examples

1. Find the degree of each vertex.



Ask students to give the degree of each vertex and write that degree near the vertex. The answers are A: 3; B: 3; C: 3; D: 3.

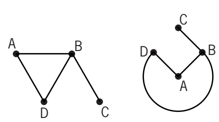
2. Explain why the graph is not traversable.

The degree of each of the four vertices is odd. A graph is traversable if exactly two of the vertices are odd or if the degrees are all even. The graph is not traversable.

Expand Their Horizons

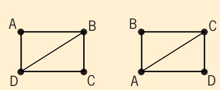
In Section 2, students will learn to identify equivalent graphs. Organization is the key to success in this section. Students must be able to identify the connections (edges) between vertices and compare those connections to the connections of other graphs. Encourage students to take their time and to review each step of their work to avoid careless errors.

To reinforce the idea that equivalent graphs can appear different, describe the connections of a graph to the class and have students draw graphs. For example, ask students to draw a graph with four vertices: A, B, C, and D. A should be connected to B and D; B should be connected to A, C, and D; C should be connected to B; and D should be connected to A and B. Remind students that edges can be curved. Ask students to come to the chalkboard to draw their graphs. Note that all of the graphs are equivalent even though they may appear different. Two sample graphs are shown below.



Common Error Alert

Students may decide that two graphs are equivalent without carefully checking all vertices and edges. For example, they may think the graphs shown below are equivalent. Point out that the graphs shown below are not equivalent because one has edge BD and the other has edge AC.



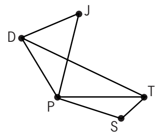
Connections

Euler circuits and Hamiltonian circuits are important topics because they require the starting vertex and the ending vertex of a path be the same. This requirement is part of many practical applications because someone who wants to minimize the distance for a route often must return to the starting point.

A snowplow operator may want to enter a particular neighborhood, clear all the streets in that neighborhood by traveling the shortest possible route, and then leave from the point of entry. The shortest route for the snowplow operator in this case is an Euler circuit. The requirement is that each edge (section of street) is traveled exactly once. Note, in this example, vertices (street intersections) may be used more than once. Other real world examples of Euler circuits include mosquito spray truck routes, highway litter pickup, and National Park trail clearing.

A salesperson may want to leave home, drive the shortest possible route to visit several cities, and then return home. The shortest route for the salesperson in this case is a Hamiltonian circuit. In this example, the vertices are the cities, and the roads are the edges. The requirement is that each vertex (city) is visited exactly once. Note, there may be some edges (connecting roads between cities) which are not used at all. Other real world examples of Hamiltonian circuits include airline flights between cities that return the pilot back to their home city, package delivery routes that return the truck back to the substation, travel itineraries between cities that return the traveler back to their home city, and campaign election stops that return the politician back to their home town.

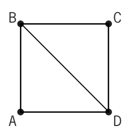
The graphs in the lesson example can be compared in a different manner. Students should list the connections found in the first graph:

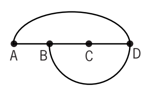


Dave to Jo, Ted, and Pat
Jo to Dave and Pat
Pat to Dave, Jo, Ted, and Sue
Ted to Dave, Pat, and Sue
Sue to Pat and Ted

Students can then compare these connections to the connections in the second graph. By comparing the connections, they will see that the graphs are equivalent.

3 **Determine if the graphs are equivalent graphs.**





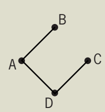
Encourage students to write down all the connections in the graphs to compare. Students should check connections carefully to avoid errors. These graphs are equivalent because all the connections match.

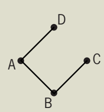
Look Beyond

Further study of graph theory will address *Euler circuits and Hamiltonian circuits*. An *Euler circuit* is a path that uses every ***edge*** of a graph exactly once and ends at the same vertex at which it begins. In other words, an Euler circuit is a traversable path that ends at the same vertex at which it begins. A *Hamiltonian circuit* is a path that uses every ***vertex*** of a graph exactly once and ends at the same vertex at which it begins.

Additional Examples

1. Are the following graphs equivalent?





While the second graph looks like the first, the connections are not the same. In the second graph, C is connected to B instead of D. The graphs are not equivalent.

2. Redraw the graph on the right in Additional Example 1 without changing the positions of the vertices so that the two graphs are equivalent. Change only the edges, keeping the vertices in the same position.