20.3 teacher notes

Objectives

- **Find the probability of independent events.**
- **Find the probability of dependent events.**

Vocabulary

Independent events
Dependent events
Probability of an event (Lesson 20-2)
Outcome (Lesson 20-2)

Prerequisites

Finding experimental probability

Finding theoretical probability

Multiplying fractions

Reducing fractions

Get Started

- Separate the class into two groups; call them Group A and Group B.
- Give each group a bag that contains two red chips and two black chips (checkers). (Small painted cubes could also be used. Other colors could be used. Other items may be used, but they must be identical, except for color.)
- In each group, instruct students to take turns drawing two chips from the bag, one at a time. In Group A, tell the students to put the first chip back into the bag before drawing the second chip. In Group B, tell the students to leave the first chip out of the bag before drawing the second chip. The two draws are considered one trial. Tell students to shake the contents of the bag before each draw.
- Instruct both groups to complete 50 trials, recording the number of trials that resulted in two red chips being drawn and the total number of trials. Then, each group should find their experimental probability of drawing two red chips.

- Bring the class back together. Say, "Group A returned their first chip to the bag before drawing the second, and that they found the probability of ***independent events***. Group B left the first chip out of the bag before drawing the second, and they found the probability of ***dependent events***."
- Ask students which group they think had a higher probability of drawing two red chips. Students should explain their reasoning. Then instruct a member from each group to write their experimental probability on the board. Were their hypotheses correct?
- Tell the class they will learn more about the probabilities of independent and dependent events in today's lesson, and the probabilities they found will be used later in the lesson.

Section 1

Expand Their Horizons

In Lesson 20-2, students learned to find probabilities of simple events, i.e., events consisting of a single outcome. In the current lesson, 20-3, students will learn to find the probability of compound events. Compound events are made up of two or more simple events. In Section 1, students will find probabilities for compound events in which the simple events are independent. In Section 2, the compound events will be made up of dependent events.

Two events are independent if the occurrence of one event does not affect the probability of the other. The classic case of selecting two cards from a deck of cards, replacing the first before selecting the second, is the first example used in the lesson. Because the first card is replaced, the same cards are available for each draw. In other words, the second draw is not affected by the result of the first draw. Another example of a pair of independent events is a pair of outcomes when two dice are rolled. The outcome on one die does not affect the outcome on the second die.

The lesson contains examples involving cards and dice. (Review with the class what cards are in a standard deck and have a deck available.) Remove the jokers from the deck of cards so that the deck contains 13 cards of each of 4 suits: hearts, diamonds, clubs, and spades. Be aware that the use of cards or dice may not be acceptable in some belief systems.

The probability that two or more independent events will occur is equal to the product of the probabilities of the individual events. For two independent events A and B, P(A and B) = P(A) · P(B). The lesson only shows calculations for two events, but the rule can be extended for more than two events. For example, P(A and B and C) = P(A) · P(B) · P(C). Some problems in the Independent Practice and Additional Practice require students to find the probability of three independent events.

In probability theory, an activity is often called an experiment. Consider the experiment of rolling two dice. Find the probability of rolling double sixes. The probability of rolling double sixes is the probability of rolling a six on the first die $\left(\frac{1}{6}\right)$ times the probability of rolling a six on the second die $\left(\frac{1}{6}\right)$. Therefore, P(6 and 6) $= \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$. Another way of showing students the correct probability is to write the sample space for rolling two dice. The sample space for an experiment is a list of all possible outcomes of the experiment.

For rolling two dice, the sample space is the following:

(1,1) (1,2) (1,3) (1,4) (1,5) (1,6)
(2,1) (2,2) (2,3) (2,4) (2,5) (2,6)
(3,1) (3,2) (3,3) (3,4) (3,5) (3,6)
(4,1) (4,2) (4,3) (4,4) (4,5) (4,6)
(5,1) (5,2) (5,3) (5,4) (5,5) (5,6)
(6,1) (6,2) (6,3) (6,4) (6,5) (6,6).

The first element in each ordered pair is the outcome on the first die, and the second element in the ordered pair is the outcome on the second die. For the event "rolling double sixes," the outcome (6, 6) is the only favorable outcome of the 36 total outcomes; so, $P(6 \text{ and } 6) = \frac{1}{36}$.

Some students may not be familiar with the word 'die' used as the singular of dice. It may be necessary to discuss this use of the word at this point.

1 **A coin is tossed, and a fair die is rolled. Find the probability of getting heads and rolling an even number.** The events are independent because the outcome of a coin toss has no affect on the probability of any outcome of a die roll and vice versa. A coin toss has two possible outcomes: heads or tails. The favorable outcome is heads. The probability of heads is $\frac{1}{2}$. Rolling a die has six possible outcomes: 1, 2, 3, 4, 5, and 6. The favorable outcomes are the even numbers: 2, 4, and 6. The probability of an even number is $\frac{3}{6}$ or $\frac{1}{2}$. The product of the probabilities is $\frac{1}{2} \cdot \frac{1}{2}$ or $\frac{1}{4}$. Therefore, the probability of getting a head and an even number is $\frac{1}{4}$ or 25%.

Outside of the classroom, students will rarely, if ever, need to toss coins and roll a die at the same time. They will, on the other hand, be likely to encounter standardized tests, which usually include multiple-choice questions. Consider the following problem that is similar to one of the problems in the lesson:

Common Error Alert

Students may think the probability of two events is $\frac{1}{2}$ when the probability of each event is $\frac{1}{2}$. Remind them to multiply probabilities. For the problem above, you may also want to explain the probability by writing the sample space for the experiment. In the sample space below, the three highlighted outcomes are favorable. Because there are 12 possible outcomes, the probability of getting a favorable outcome is $\frac{3}{12} = \frac{1}{4}$.

H1 **H2** H3 **H4** H5 **H6**
T1 T2 T3 T4 T5 T6

Suppose a student runs out of time and guesses on the last three items of a test. If the choices are A, B, C, D, and E, what is the probability that the student gets all three answers correct? The probability of selecting the correct answer to any individual question is $\frac{1}{5}$ because there is one right answer out of five possible answers. Because each question's guess is independent of the other guesses, the probability of correctly answering all three questions is the product of the probabilities of guessing correctly on each question or $\frac{1}{5} \cdot \frac{1}{5} \cdot \frac{1}{5}$, which equals $\frac{1}{125}$.

2 **Two letters from the word "apple" are selected at random *with replacement*. What is the probability of selecting two "p's"?** Consider the letters in the word to be written on cards, $\boxed{a}\ \boxed{p}\ \boxed{p}\ \boxed{l}\ \boxed{e}$, and placed in a bag. The events are independent because the first letter is being replaced before selecting the second letter. The probability of selecting a "p" each time is $\frac{2}{5}$ because there are two ways of getting a favorable outcome, $\boxed{p}\ \boxed{p}$, out of the five possible choices. Therefore, the probability of selecting two "p's" in this situation is $\frac{2}{5} \cdot \frac{2}{5} = \frac{4}{25}$ or 16%.

Additional Examples

1. Records of a certain delivery service show that 6% of its packages arrive late. If Dan and Rob both use this delivery service, what is the probability that both of their packages arrive late? Express the answer as a fraction and as a percent.

P(both late) $= \frac{6}{100} \cdot \frac{6}{100} = \frac{36}{10{,}000} = \frac{9}{2{,}500}$
$= 0.0036 = 0.36\%$
The probability that both their packages arrive late is $\frac{9}{2{,}500}$ or 0.36%.

2. Suppose that 80% of all the cars used by a car rental service have a CD player. Joe plans to rent a car from the service for three different trips. What is the probability that Joe will get a CD player for all three trips? Express the answer as a percent.

$80\% = 0.80 = 0.8$
$(0.8)(0.8)(0.8) = 0.512 = 51.2\%$
The probability that Joe will get a CD player for all three trips is 51.2%.

Section 2

Expand Their Horizons

When an experiment involves making selections, such as drawing marbles from a bag, a selection can be replaced, or it can be kept. If all selections are replaced, then events in the experiment are independent. If a selection is not replaced before making the next selection, the events involved are dependent because the number of available items for the next selection is reduced by one, and therefore, the probability of getting a particular result on the next selection is affected.

In Section 2, students will find probabilities of dependent events. As with independent events, the probability of dependent events is found by multiplying the individual probabilities of the events. However, when two events are dependent, the probability of the second event is affected by the occurrence of the first event.

Suppose a bag has five marbles: two red and three blue. Find the probability of randomly selecting a red marble and then, another red marble.

	With replacement (Independent events)
First selection	P(R) $= \frac{2}{5}$
Second selection	P(R) $= \frac{2}{5}$
	P(R, then R) $= \frac{2}{5} \cdot \frac{2}{5} = \frac{4}{25} = 12\%$

	Without replacement (Dependent events)
First selection	P(R) $= \frac{2}{5}$
Second selection	P(R) $= \frac{1}{4}$ (4 marbles remain after the first selection, and 1 of them is red)
	P(R, then R) $= \frac{2}{5} \cdot \frac{1}{4} = \frac{2}{20} = \frac{1}{10} = 10\%$

Now suppose a bag has the same five marbles: two red and three blue. Find the probability of randomly selecting a red marble and then, a blue marble.

	With replacement (Independent events)
First selection	$P(R) = \frac{2}{5}$
Second selection	$P(B) = \frac{3}{5}$
	$P(R, \text{ then } B) = \frac{2}{5} \cdot \frac{3}{5} = \frac{6}{25} = 24\%$

	Without replacement (Dependent events)
First selection	$P(R) = \frac{2}{5}$
Second selection	$P(B) = \frac{3}{4}$ (4 marbles remain after the first selection, and 3 of them are blue)
	$P(R, \text{ then } B) = \frac{2}{5} \cdot \frac{3}{4} = \frac{6}{20} = \frac{3}{10} = 30\%$

The examples in the tables above show that when events are dependent, the probability of the second event is affected by the occurrence of the first event. The probability of a compound event made up of dependent events A and B is written as P(A then B) = P(A) · P(B after A). The lesson only shows calculations for two events, but the rule can be extended for more than two events. For example, P(A then B then C) = P(A) · P(B after A) · P(C after B and A). Some problems in the Independent Practice and Additional Practice require students to find the probability of three dependent events.

In the lesson, the DVD characters find the probability of drawing two hearts from a deck of cards. The first card is not replaced. Because 13 of the 52 cards are hearts, the probability of drawing a heart on the first draw is $\frac{13}{52}$ or $\frac{1}{4}$. After the first card is drawn, the deck of cards has been changed. If the first card drawn is a heart, then only 12 of the remaining 51 cards are hearts. So, the probability of drawing a heart the second time is $\frac{12}{51}$ or $\frac{4}{17}$. The probability of drawing two hearts without replacement (dependent events) is $\frac{1}{4} \cdot \frac{4}{17} = \frac{4}{68} = \frac{1}{17}$. Remind the students that earlier in the lesson, they found that the probability of drawing two hearts with replacement (independent events) was $\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$. Instruct students to convert these probabilities to percents and to compare them ($\frac{1}{17} \approx 5.9\%$ and $\frac{1}{16} \approx 6.3\%$). It should make sense to the students that there is a higher probability of drawing two hearts with replacement because there is one more heart in the deck for the second draw when the first card is replaced.

Recall the opening activity. The theoretical probability of selecting two red chips with replacement (independent events) is $\frac{2}{4} \cdot \frac{2}{4} = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. Group A found the experimental probability for this situation. The theoretical probability of selecting two red chips without replacement (dependent events) is $\frac{2}{4} \cdot \frac{1}{3} = \frac{2}{12} = \frac{1}{6}$. Group B found the experimental probability for this. The theoretical probability is lower for Group B; because once a red chip is selected and not replaced, there are fewer red chips remaining in the bag for the second selection. The second chip was selected from a bag which contained only one red chip and two black ones, a total of three. Students should find these theoretical probabilities on their own after they have viewed the lesson. Because the experimental probabilities found by the groups will most likely differ from the theoretical probabilities just discussed, it may be necessary to review the difference between theoretical and experimental probabilities.

3 **The yearbook staff randomly picks pictures from a box for a slide show presentation for the senior class. The box contains 25 color photos and 50 black and white photos. What is the probability of getting a black and white photo and then a color photo if the first photo is not replaced?** It must be assumed the 75 photos are randomly mixed in the box, and the person picking the pictures cannot see them. Because there are 50 black and white photos in the box containing a total of 75 photos, the probability that the first photo chosen is black and white is $\frac{50}{75}$ or $\frac{2}{3}$. Since the first photo is removed, there are now only 74 photos left in the box. All the color photos are still in the box because the one removed is assumed to be black and white.

Therefore, the probability that the second selection is a color photo is $\frac{25}{74}$; because once a black and white photo is removed, there are 25 color photos out of 74 photos left in the box. $\frac{50}{75} \cdot \frac{25}{74} = \frac{1{,}250}{5{,}550} = \frac{125}{555} = \frac{25}{111}$. The probability of selecting a black and white photo and then a color photo is $\frac{25}{111}$, which is approximately 23%.

4 **Two letters from the word "apple" are selected at random *without replacement*. What is the probability of selecting two "p's"?** This is the same situation as was described in Problem 2 in the Guided Notes, except that the first letter drawn is *not* replaced. So, these events are dependent. The probability of selecting the first "p" is $\frac{2}{5}$. When the first "p" is not replaced, there is only one "p" and a total of four letters left; so, the probability of selecting the second "p" is $\frac{1}{4}$. Multiply the probabilities. $\frac{2}{5} \cdot \frac{1}{4} = \frac{2}{20} = \frac{1}{10}$ or 10%. Compare this to the answer found in Problem 2 of the Guided Notes. $\left(\frac{4}{25} \text{ or } 12\%\right)$

Common Error Alert

Students may use the wrong numbers to form a probability. Encourage them to determine whether the first item selected is replaced. Guided Notes Problems 2 and 4 should be compared to illustrate the importance of the wording.

There are many real world situations which require finding the probability of dependent events. For example, quality-control managers make decisions based on such probabilities. Because it is too time-consuming to test every single object that passes through an assembly line, it is often the case that only a few are tested. Based on probability, rules are made to determine whether to accept or reject a batch of manufactured items.

Suppose a company produces batteries. From each lot of 100 batteries, two are randomly selected, without replacing the first battery before selecting the second. If both are defective, the entire lot is rejected. Suppose a lot actually has 35 defective batteries. Find the probability that the two batteries selected from this lot for testing will be defective. The probability that the first battery is defective is $\frac{35}{100}$ because there are 35 defective batteries out of a total of 100 batteries. The probability that the second battery is defective is $\frac{34}{99}$; because once the first defective battery is removed, there are only 34 defective batteries left in a lot of 99 batteries. $\frac{35}{100} \cdot \frac{34}{99} = \frac{1{,}190}{9{,}900}$ or approximately 12%. The owner of the company would almost certainly say that a lot of 100 batteries with 35 defective should be rejected. However, based on the current procedure, there is only a 12% chance that two of the 35 defective batteries will be tested and that the lot will be rejected. Is this acceptable? Should more stringent standards be set? These are questions that quality control managers must address.

Look Beyond

Probability is being used in a growing number of fields, and many college majors, including those in business, psychology, and education, now need to take at least one probability and statistics course. It is beneficial for students to learn probability now, so they can build upon and extend their knowledge in later courses. Probability is also used in everyday life. For example, weather forecasts often use probabilities. Informal estimates of probabilities are used in many decisions. For example, a person may decide whether or not to buy a raffle ticket based on an informal estimate of the probability of winning a prize. And if more gamblers knew about probability, they would stay away from the casinos!

Additional Examples

1. **Julie has 50 CDs in her CD collection. She knows that 10 are scratched, but she doesn't remember which 10 they are. What is the probability that Julie selects a CD that is not scratched and without replacing it, selects another that is not scratched? Express the answer as a fraction, and as a percent rounded to the nearest tenth of a percent.**

   If 10 are scratched, then 40 are not scratched. Because the first CD is not replaced (dependent events), the denominator in the second fraction is reduced to 49.
   P(not scratched then not scratched)
   $= \frac{40}{50} \cdot \frac{39}{49} = \frac{4}{5} \cdot \frac{39}{49} = \frac{156}{245} \approx 63.7\%.$
   The probability that Julie selects a CD that is not scratched, followed by another that is not scratched, is $\frac{156}{245}$ or about 63.7%.

2. **A quality control worker at an orange-juice manufacturing plant rejects batches of oranges when four randomly selected oranges, chosen without replacement from a crate of 50, are rotten. If a crate of 50 oranges contains 18 rotten oranges, what is the probability that the crate will be rejected? Express the answer as a percent rounded to the nearest tenth of a percent.**

   Extend the rule for finding the probability of two dependent events to finding the probability of four dependent events.
   $P(r) = P(\text{rotten orange})$
   $P(r \text{ then } r \text{ then } r \text{ then } r) =$
   $\frac{18}{50} \cdot \frac{17}{49} \cdot \frac{16}{48} \cdot \frac{15}{47} = \frac{73{,}440}{5{,}527{,}200} \approx 0.0133 \approx 1.3\%$
   There is a 1.3% probability that the crate will be rejected.