20.1 teacher notes

Objectives

- Use a tree diagram.
- Use the Fundamental Counting Principle.
- Evaluate factorials.
- Find permutations and combinations.

Vocabulary

Tree diagram
Fundamental Counting Principle
Arrangement
Factorial
Permutation
Combination

Prerequisites

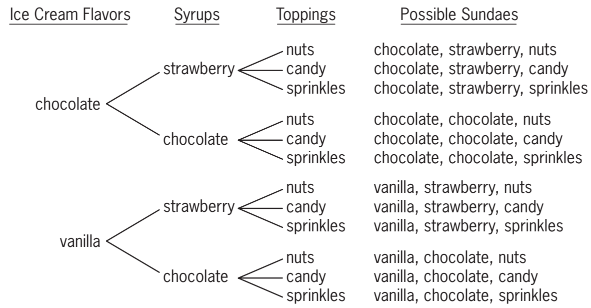
Multiplying and dividing whole numbers

Canceling common factors to reduce fractions

Get Started

- Ask each student to number a piece of paper from one to four.
- Say, “Please use the numbers to rank the four seasons from your favorite to your least favorite. Choose from spring, summer, winter, and fall. Write your favorite season first.” Give students a moment to complete the task, ask several students to disclose their lists, and then write them on the board. Do not write any list more than once.
- When there are three different lists on the board, say, “The lists on the board are all different. How many different lists do you think are possible?” Do not reveal the answer (24) but let students try to think of ways to find the answer.
- Pose the additional question “What if I had asked you to list only your top two favorite seasons? How many different lists could have been made?” Again, do not reveal the answer (12) but encourage discussion. Point out that, for example, the list *spring* and *summer*, is different from the list *summer* and *spring*.

total of 2 · 2 possibilities. For the third part of the sundae, there are three choices of toppings. On the tree diagram, list the three choices next to each of the choices of syrup.



There are 12 sundaes possible; they can be listed next to the branches of the tree diagram, as shown in the diagram. Using the Fundamental Counting Principle, there are 2(flavors of ice cream) · 2(kinds of syrup) · 3(toppings) or 12 possible sundaes.

Because tree diagrams enhance the students' understanding of the Fundamental Counting Principle, they should be used whenever the numbers are small enough.

1 **Suppose a meal consists of an appetizer, an entrée, and a dessert. Find the total number of different meals from which you can choose if there are five appetizers, three entrées, and six desserts.** To find the total number of possible meals where a diner can choose exactly one appetizer, one entrée, and one dessert, find the product of the numbers of options for each course. Since $5 \cdot 3 \cdot 6 = 90$, there are 90 possible meals. Point out to students that a tree diagram is impractical for this problem.

Additional Examples

1. A restaurant offers a "Pick 3 Combo" lunch. Diners can choose from four soups, two salads, and three beverages. How many different meals are possible?

Use the Fundamental Counting Principle to find the number of different meals that are possible.

$\_\_\_\_\_\_ \cdot \_\_\_\_\_\_ \cdot \_\_\_\_\_\_$

Soup Salad Beverage

$$4 \cdot 2 \cdot 3 = 24$$

There are 24 possible "Pick 3 Combo" lunches.

2. Of 15 employees, a store manager must assign one employee to work the stockroom, one employee to work the cash register, and one employee to greet customers. How many ways are there to choose three employees from the 15 employees?

There are 15 ways to choose an employee to work the stockroom. When that is done, there are 14 ways to choose an employee to work the cash register, then 13 ways to choose a greeter.

$\_\_\_\_\_\_ \cdot \_\_\_\_\_\_ \cdot \_\_\_\_\_\_$

Stock Register Greeter

15 choices for stockroom

14 choices for register

13 choices for greeter

$$15 \cdot 14 \cdot 13 = 2{,}730$$

There are 2,730 ways to choose three employees from 15 employees.

Section 2

Expand Their Horizons

In Section 2, students will be introduced to the word "factorial" and the symbol which represents it, "!." $n! = n(n-1)(n-2)\ldots(3)(2)(1)$. In words, to find n factorial, find the product of the integers 1 to n. So, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$. (Note that $n!$ is defined only when n is a positive integer, with $0!$ defined to be 1; i.e., $0! = 1$.)

Suggest that students explore finding factorials on their calculators. Most scientific calculators have a factorial key ($\boxed{n!}$ or $\boxed{x!}$) On the TI-83 graphing calculator, the factorial function can be found by pressing the $\boxed{\text{MATH}}$ key, going to the PRB (probability) tab, and selecting Option 4.

Factorials are used when calculating combinations and permutations. A *permutation* is an arrangement of objects in which *order matters*. That is, the arrangement AB is considered to be different from the arrangement BA. A *combination* is an arrangement of objects in which *order does not matter*. In a combination, the arrangement AB is not distinguished from the arrangement BA.

One example of a permutation follows: Suppose a service club has six members. If the names of two members are pulled from a hat, the first one being assigned the job of president and the second being assigned the job of treasurer, the arrangement of the names is a *permutation*. In this case, order matters, since the possibility *Steve is president and Mary is treasurer* is different from the possibility *Mary is president and Steve is treasurer*. To find the number of possibilities for president and treasurer, use the formula ${}_nP_r = \frac{n!}{(n-r)!}$:

$${}_6P_2 = \frac{6!}{(6-2)!} = \frac{6!}{4!} = \frac{6 \cdot 5 \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = \frac{6 \cdot 5}{1} = 30.$$

A shorter way to do this problem is

$${}_6P_2 = \frac{6!}{(6-2)!} = \frac{6!}{4!} = \frac{6 \cdot 5 \cdot \cancel{4!}}{\cancel{4!}} = \frac{6 \cdot 5}{1} = 30.$$

So, there are 30 permutations of two members from the six club members.

Consider the difference between a permutation and a combination. Suppose that in the same service club, two members are randomly selected to represent the group at an award ceremony. In this case, order does not matter, and any two members selected forms a *combination*. That is, the combination *Henry and Beth* is the same as the combination *Beth and Henry*. To find the number of ways that two members can be chosen, use the formula ${}_nC_r = \frac{n!}{r!(n-r)!}$:

$${}_6C_2 = \frac{6!}{2!(6-2)!} = \frac{6!}{2!4!} = \frac{6 \cdot 5 \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{2 \cdot 1 \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = \frac{30}{2} = 15,$$

or more efficiently:

$${}_6C_2 = \frac{6!}{2!(6-2)!} = \frac{6!}{2!4!} = \frac{6 \cdot 5 \cdot \cancel{4!}}{2 \cdot 1 \cdot \cancel{4!}} = \frac{30}{2} = 15.$$

So, there are 15 combinations of two members from the six club members.

As shown in the examples above, there is much cancellation of common factors that can be done to simplify the arithmetic when evaluating factorials. Canceling factorials saves time.

2 **A band director must choose five drummers, out of nine, to march in a parade. In how many different ways can the director line up the five drummers, choosing from nine drummers?** First, determine whether the arrangement described is a permutation or a combination. In this exercise, the key phrase is "line up." If the drummers are numbered from one to nine, then the line-up 12345 is different from the line-up 54321. So, order matters, and the permutation formula is needed. In this example, $n = 9$ and $r = 5$, so evaluate ${}_9P_5$. There are ${}_9P_5 = \frac{9!}{(9-5)!} = \frac{9!}{(4)!} = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 15{,}120$, or more efficiently: ${}_9P_5 = \frac{9!}{(9-5)!} = \frac{9!}{(4)!} = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot \cancel{4!}}{\cancel{4!}} = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 =$ 15,120 ways to line up five of the nine drummers.

- Finally, say, "In both situations that we have considered so far, the order of the seasons mattered. Now let's think about a third list. What if I had simply asked you to each name your two favorite seasons? How many different possibilities are there?" As students think, point out that a student writing *fall*, *winter* is naming the same two seasons as a student writing *winter*, *fall*; so, for this list, order does not matter.
- Say, "Today we will learn about *permutations* and *combinations*. These are mathematical words to describe different ways to choose and arrange objects, like the names of the seasons. We will learn how to determine the possible number of ways to choose and to arrange some or all of the objects in a set."

Section 1

Expand Their Horizons

In this lesson, students will be introduced to the Fundamental Counting Principle, permutations, and combinations.

Many real-life situations involve making several choices, each from a list of possible options. For example, an ice cream shop may offer sundaes consisting of chocolate or vanilla ice cream (two ice cream choices), strawberry or chocolate syrup (two syrup choices), and nuts, candy, or sprinkles (three topping choices).

The Fundamental Counting Principle states:

If there are *m* ways to make the first choice, and *n* ways to make the second choice, then there are $m \cdot n$ ways to make the two choices.

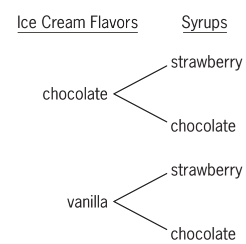
It is important to note that the Fundamental Counting Principle is stated using only two choices. That is, two choices must be made: one from a list of m options, another from a list of n options. However, this principle can be applied for any number of choices. To find the total number of possibilities, simply find the product of the numbers of options for each choice. So, in the ice cream sundae example above, there are $2 \cdot 2 \cdot 3 = 12$ possible different sundaes that can be made.

A tree diagram can be used to illustrate why the Fundamental Counting Principle works. In the example above, there are two choices of ice cream. List them vertically.

chocolate

vanilla

There are two choices of syrup; the kinds of syrup can be put on either choice of ice cream, so the two choices of syrup will be listed next to the two choices of ice cream.



Because there are two choices of ice cream, there are two entries in the first column of the diagram. Because there are two choices of syrup, there are two entries branching from each of the ice cream choices. This gives a

3 **The Mr. Smoothie's shop has five different types of fresh fruit available. A Supreme smoothie is a blend of three different fruits. How many different supreme smoothies are possible?** Since the order that the three fruits are added to the drink doesn't matter because they all get mixed together, each smoothie represents a combination. Evaluate ${}_nC_r$, using $n = 5$ and $r = 3$: ${}_5C_3 = \frac{5!}{3!(5-3)!} = \frac{5!}{3!2!} = \frac{5 \cdot 4 \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{3} \cdot \cancel{2} \cdot \cancel{1} \cdot 2 \cdot 1} = \frac{20}{2} = 10$. There are 10 possible combinations when three of five fruits are chosen, so 10 different supreme smoothies can be made.

Look Beyond

Finding permutations and combinations is important in finding probabilities. For example, suppose 100 raffle tickets are sold, and there will be two winners. Also, suppose a student and his friend each have one of the raffle tickets, and each of the 100 tickets has an equal chance of being chosen. What is the probability the student and his friend will both win?

First, find the number of ways that two tickets can be chosen from 100 tickets, where order does not matter.

${}_{100}C_2 = \frac{100!}{2!(100-2)!} = \frac{100!}{2!98!} = \frac{100 \cdot 99}{2 \cdot 1} = \frac{9{,}900}{2} = 4{,}950.$

There are 4,950 ways to choose two tickets from 100 tickets, so the probability that the student and his friend will both win is $\frac{1}{4{,}950}$ or about 0.02%.

Additional Examples

1. Caroline can fit four textbooks in her backpack. If she has eight textbooks, how many different four-textbook loads are possible?

Since the order of the textbooks does not matter, use the formula for combinations.

$$
\begin{aligned}
{}_8C_4 &= \frac{8!}{4!(8-4)!} \\
&= \frac{8!}{4!(4)!} \\
&= \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{(4 \cdot 3 \cdot 2 \cdot 1)(\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1})} \\
&= \frac{\cancel{8}^2 \cdot 7 \cdot \cancel{6} \cdot 5}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} \\
&= \frac{2 \cdot 7 \cdot 5}{1} \\
&= 70
\end{aligned}
$$

There are 70 different textbook loads.

2. There are nine runners competing in a race. How many different outcomes are possible for 1st, 2nd, and 3rd place?

Find the number of lists of three runners possible when there are nine runners. Since the order in which they are listed matters, use the formula for permutations.

$$
\begin{aligned}
{}_9P_3 &= \frac{9!}{(9-3)!} \\
&= \frac{9!}{6!} \\
&= \frac{9 \cdot 8 \cdot 7 \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}} \\
&= 9 \cdot 8 \cdot 7 \\
&= 504
\end{aligned}
$$

There are 504 different outcomes for 1st, 2nd, and 3rd place.