18.4 teacher notes

Objectives

- Recognize distance as the absolute value of a difference.
- Demonstrate the correct use of the Pythagorean Theorem.
- Use the distance formula to solve problems.
- Use the midpoint formula to solve problems.

Vocabulary

Pythagorean Theorem (Lesson 18-3)
Hypotenuse (Lesson 18-3)
Absolute value (Lesson 1-2)
Coordinate plane (Lesson 7-1)
Distance formula
Midpoint
Midpoint formula
Diameter

Prerequisites

Evaluating expressions using order of operations

Using formulas

Graphing ordered pairs on the coordinate plane

Get Started

- Give each student a piece of graph paper.
- Tell the class to plot the points (3, 3) and (7, 6). Say, "Label the point (3, 3) as Bill's house. Label the point (7, 6) as Sara's house. Imagine that the graph paper is a map. Each square on the graph paper represents a city block, and the gridlines are roads."
- Say, "Bill is going to walk to Sara's house. He can only walk due north, south, east, and west. He cannot cut diagonally through any block since the inside of each block is filled with buildings."
- Ask, "What is the shortest distance Bill can walk to get to Sara's house?" Give students a moment to determine the correct answer is seven blocks.
- Next, tell students that Fleigle plans to fly from Bill's house to Sara's house. Because she can fly over the buildings, Fleigle is free to fly in any direction.
- Ask, "What is the shortest path Fleigle can fly to get to Sarah's house?" (The shortest distance between two points is a straight line.) Have students draw the line segment representing Fleigle's path on their graph paper. Next, ask them to estimate the length of the segment.

Estimations will vary. Point out that estimations less than four and greater than seven are not reasonable.

- Direct students to test their estimates by cutting a strip of graph paper to use as a ruler. This activity should convince the students the distance is five units.

Section 1

Expand Their Horizons

In this lesson, two important formulas from analytic geometry are introduced. The *distance formula* is used to find the length of a line segment with given endpoints; the *midpoint formula* is used to find the point on a segment that is equidistant from each of its endpoints.

The lesson opens with a review of the Pythagorean Theorem. Students should be familiar with this theorem from previous courses. In any right triangle, the sum of the squares of the lengths of the legs is equal to the square of the length of the hypotenuse. In symbols $a^2 + b^2 = c^2$, a and b are the lengths of the legs, and c is the length of the hypotenuse. Review the vocabulary of right triangles. The *legs* are the sides of the triangle that meet at a right angle. The *hypotenuse* is the remaining side and lies opposite the right angle.

To find the distance between two points on a number line, find the absolute value of their difference. This is a simple idea hidden behind complex language and may be better understood visually than verbally. For example, to find the distance from –4 to 2, use the formula $|-4 - 2|$ or imagine the distance from –4 to 0 (four units) plus the distance from 0 to 2 (two units). Either method shows that the distance is six units.

On the coordinate plane, distances can be found using the number line method for vertical and horizontal line segments. For vertical segments, the y-axis is the number line, and the absolute value of the difference of the y-coordinates $(y_2 - y_1)$ is the length. For horizontal segments, the x-axis is the number line, and the difference of the x-coordinates $(x_2 - x_1)$ is the length.

Distances that are neither horizontal nor vertical can be found using the Pythagorean Theorem. A vertical and a horizontal line segment with a common endpoint will form the two legs of a right triangle on the coordinate plane. Solve the formula $a^2 + b^2 = c^2$ for the length of the hypotenuse: $\sqrt{a^2 + b^2} = c$. Here, a and b are the lengths of the legs. If the vertices of two points are known, then the distance between these points is given by $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = c$. Here, (x_1, y_1) and (x_2, y_2) are the coordinates of the points.

Common Error Alert

Students may assign the wrong coordinates to the variables in the distance formula. Ask students to take a moment before starting the computation to write "x_1," "y_1," "x_2," and "y_2" above the ordered pairs next to the appropriate coordinates. Also, instruct them to write out the distance formula each time it is used.

1 To find the distance between (4, –6) and (–4, –10), first assign variables to the coordinates $x_1 = 4$, $y_1 = -6$, $x_2 = -4$, and $y_2 = -10$. Substitute these values into the distance formula and simplify. Because the distance from (4, –6) to (–4, –10) is the same as the distance from (–4, –10) to (4, 6), it is also valid to write

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} =$$
$$\sqrt{(4 - (-4))^2 + (-6 - (-10))^2} =$$
$$\sqrt{(8)^2 + (4)^2} = \sqrt{64 + 16} = \sqrt{80} =$$
$$4\sqrt{5}.$$

Common Error Alert

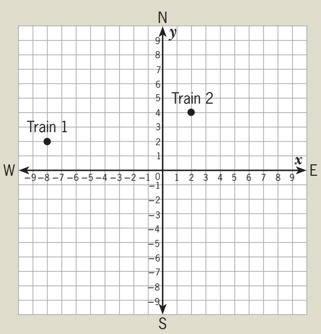
Students may fail to include the negative sign on negative coordinates. For Guided Notes Question 1, they may write $d = \sqrt{(-4 - 4)^2 + (10 - 6)^2}$. Remind them that inside each set of parentheses in the distance formula, a *difference* is taken. When a negative number is subtracted, the subtraction becomes addition of the opposite. If this is a persistent problem for some students, encourage students to write $d = \sqrt{(\_\_ - \_\_)^2 + (\_\_ - \_\_)^2}$ before substituting the coordinates into the formula.

Students can be overwhelmed by the appearance of the distance formula. Such students may prefer a more visually-oriented approach to finding distance. For example, to find the distance from (4, –6) to (–4, –10), students should mentally determine the distance between the *x*-coordinates 4 and –4 (eight units). Then, they should place the solution into the Pythagorean Theorem equation, $\sqrt{a^2 + b^2} = c$ to show $\sqrt{8^2 + b^2} = c$. Repeat the process for the distance between the *y*-coordinates –6 and –10 (four units), giving $\sqrt{8^2 + 4^2} = c$. Then, they can simplify the radicand to find the length of the hypotenuse.

Additional Examples

1. Two commuter trains are headed into the station. Train 1 is located eight miles west and two miles north of the station. At the same time, Train 2 is located two miles east and four miles north of the station. What is the distance between the trains at that time?

Assign ordered pairs to the location of each train. Then, use the distance formula to find the distance between them.



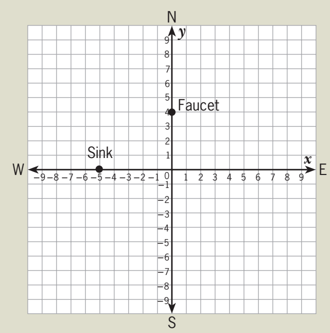
Train 1: (–8, 2) Train 2: (2, 4)

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$d = \sqrt{(2 - -8)^2 + (4 - 2)^2}$$
$$d = \sqrt{(10)^2 + (2)^2}$$
$$d = \sqrt{100 + 4}$$
$$d = \sqrt{104}$$

The distance between the trains is $\sqrt{104}$ miles or approximately 10.2 miles.

2. A plumber is fixing a sink five km west of his company's shop. His coworker is installing a faucet four km north of the shop. What is the distance between the plumbers?

In this problem, each ordered pair location lies on an axis. The plumber five km west of the shop is at (–5, 0); the plumber four km north of the shop is at (0, 4).



Sink: (–5, 0) Faucet: (0, 4)

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$d = \sqrt{(0 - -5)^2 + (4 - 0)^2}$$
$$d = \sqrt{(5)^2 + (4)^2}$$
$$d = \sqrt{25 + 16}$$
$$d = \sqrt{41}$$

The distance between the plumbers is $\sqrt{41}$ km or approximately 6.4 km.

Section 2

Expand Their Horizons

In Section 2, the midpoint formula is introduced. The midpoint of a line segment is the point on the line segment that is the same distance from one endpoint as it is from the other.

To find the midpoint of a segment, given its endpoints, form an ordered pair by averaging the x-coordinates and y-coordinates of the endpoints. The x-coordinate of the midpoint is the average of the endpoints' x-coordinates. The y-coordinate of the midpoint is the average of the endpoints' y-coordinates. Remind students the average is also referred to as the arithmetic average or the mean.

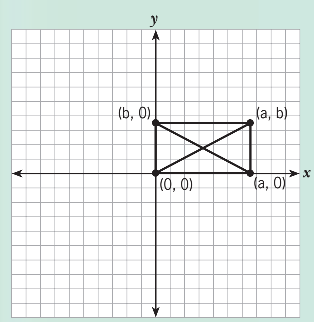
2 To find the distance between the houses, use the distance formula $x_1 = 4$, $y_1 = 2$, $x_2 = 0$, and $y_2 = -1$. Substitute into the distance formula to get $d = \sqrt{(0 - 4)^2 + (-1 - 2)^2} = \sqrt{(-4)^2 + (-3)^2} = \sqrt{16 + 9} = \sqrt{25} = 5$. The distance from Mike's house to Angelo's house is five units.

3 To find the distance between the houses, use the distance formula $x_1 = -4$, $y_1 = -4$, $x_2 = 0$, and $y_2 = -1$. Substitute into the distance formula to get $d = \sqrt{(0 - (4))^2 + (-1 - (-4))^2} = \sqrt{(4)^2 + (3)^2} = \sqrt{16 + 9} = \sqrt{25} = 5$. The distance from Mike's house to Brenda's house is five units.

Since Mike's house lies on the segment whose endpoints are Angelo's house and Brenda's house and the distances to each endpoint are the same, Mike's house is located at the midpoint of the segment with endpoints at Angelo's house and at Brenda's house.

Look Beyond

The distance and midpoint formulas are used extensively in analytic geometry, the field of mathematics that marries algebra and geometry. In analytic geometry, geometric figures are placed on the coordinate plane. One application of analytic geometry allows properties of figures to be shown and to be proven using algebra. For example, it can be **shown** the diagonals of a certain rectangle are congruent (have the same length) by plotting the rectangle on the coordinate plane; then, using the distance formula, find the lengths of its diagonals. It can be **proven** that the diagonals of a rectangle are always congruent by plotting a nonspecific rectangle on the coordinate plane (for example, with vertices (0, 0), (a, 0), (a, b), and (0, b)) and by finding the lengths of its diagonals using the distance formula.



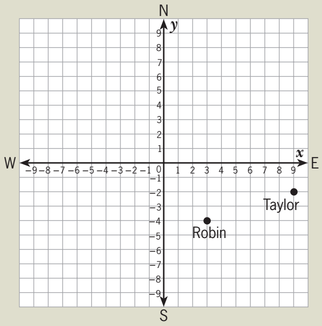
Diagonal from (0, 0) to (a, b):
$\sqrt{(a - 0)^2 + (b - 0)^2} = \sqrt{a^2 + b^2}$

Diagonal from (0, b) to (a, 0):
$\sqrt{(a - 0)^2 + (0 - b)^2} = \sqrt{a^2 + b^2}$

Additional Examples

1. Taylor lives nine miles east and two miles south of school. Robin lives three miles east and four miles south of school. Relative to the school, what is the location of Abby's house if Abby lives at a point exactly halfway between Robin and Taylor?

Assign ordered pair coordinates to Taylor's and Robin's homes. Then, use the midpoint formula to find the midpoint.



Taylor: (9, –2) Robin: (3, –4)

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$\left(\frac{9 + 3}{2}, \frac{-2 + -4}{2}\right)$$

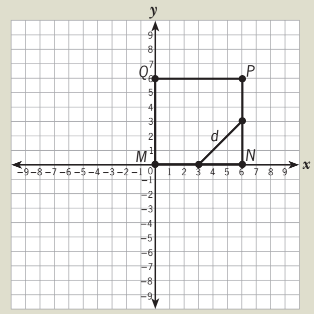
$$\left(\frac{12}{2}, \frac{-6}{2}\right)$$

$$(6, -3)$$

Abby lives six miles east and three miles south of school.

2. Square *MNPQ* has vertices at *M*(0, 0), *N*(6, 0), *P*(6, 6), and *Q*(0, 6). What is the length of the segment whose endpoints are the midpoints of sides $\overline{MN}$ and $\overline{NP}$?

First, find the midpoints of sides $\overline{MN}$ and $\overline{NP}$. Then, use those points as endpoints to find the distance. Drawing the figure on a coordinate plane may help students visualize the problem.



Midpoint of $\overline{MN}$:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$\left(\frac{0 + 6}{2}, \frac{0 + 0}{2}\right)$$

$$(3, 0)$$

Midpoint of $\overline{NP}$:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$\left(\frac{6 + 6}{2}, \frac{0 + 6}{2}\right)$$

$$(6, 3)$$

$$d = \sqrt{(6 - 3)^2 + (3 - 0)^2} = \sqrt{(3)^2 + (3)^2} = \sqrt{9 + 9} = \sqrt{18} = \sqrt{9 \cdot 2} = 3\sqrt{2}$$

The length of the segment joining the midpoints of $\overline{MN}$ and $\overline{NP}$ is $3\sqrt{2}$ units.

Manipulatives

A geoboard can be used to demonstrate the origin of the Pythagorean Theorem and its use in determining the distance between two points. Notice the *x*- and *y*-axes can be placed by drawing them on the boards or placing rubber bands to mark them for these exercises.

For a visual representation of the Pythagorean Theorem, a 3-4-5 sided right triangle can be used, and at least a 12 × 12 peg geoboard (or paper replica) will be needed. Mark the *x*- and *y*-axes so the point (7, 7) can fit in Quadrant I. The steps that follow can be demonstrated to or done by the students. Form a triangle with points (0, 0), (3, 0), and (0, 4). Discuss that this is a right triangle. Along the vertical side of the triangle, form a square with a new band; the corners of the square should be (–4, 0), (0, 0), (0, 4), and (–4, 4). Along the horizontal side of the triangle, form a square with a new band; the corners of the square should be (0, –3), (3, –3), (3, 0), and (0, 0). Along the hypotenuse, form a square with a new band; the corners of the square should be (0, 4), (3, 0), (7, 3), and (4, 7). By counting the 1 × 1 squares within the enclosed areas, find the area of the square along the vertical side of the triangle and the area of the square along the horizontal side. These should be 16 and nine square units respectively. To find the area of the square along the hypotenuse, students should trace the square on paper, cut it out, and rotate it so that its sides are parallel with those of the geoboard. Counting the 1 × 1 squares reveals the area of this square is 25 square units. Point out to students that 16 + 9 = 25; the sum of the squares of the sides equals the square of the hypotenuse.

The geoboard can help students visualize the use of the Pythagorean Theorem to find the distance between two points. Place the *x*- and *y*-axes so that the given points can be found. Place a band between them. Pull one side of the band so that a triangle is formed with a horizontal and vertical line (making a right angle). If the line segment between two points has a negative slope, the right angle will be formed by pulling the left side of the band down and to the left or by pulling the right side of the band up and to the right. If the line segment between two points has a positive slope, the right angle will be formed by pulling the left side up and to the left or by pulling the right side down and to the right. When a perfect right triangle is formed (one side runs vertically along a column of pegs and another side runs, horizontally along a row of pegs), place the band around the third point. Note, the two possible points for finding the point at the right angle can be found by interchanging the *y* values of the given points. For example, in finding the distance between (–5, 1) and (3, 7), the point that makes a right angle with this segment as the hypotenuse can be either (–5, 7) or (3, 1).

The lengths of the vertical and horizontal sides of the triangle are found by counting the spaces on those sides. These are placed as a and b in the Pythagorean Theorem, written as $c = \sqrt{a^2 + b^2}$. In effect, this performs the $(x_2 - x_1)$ and $(y_2 - y_1)$ steps of the distance formula, with the vertical distance being the change in y values and the horizontal being the change in x values. Then, the equation is solved for the distance between the two points, given here as the length of segment c.