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17.4 teacher notes

Objectives

- Divide rational expressions containing radicals.
- Simplify radical expressions using the conjugate.

Vocabulary

Product Property of Square Roots (Lesson 17-1)
Quotient Property of Square Roots
Absolute value (Lesson 1-2)
Simplest form of a radical (Lesson 17-1)
Perfect square (Lesson 17-1)
Rationalizing the denominator
Conjugate

Prerequisites

Simplifying fractions

Simplifying radicals by using the Product Property of Square Roots

Multiplying radicals

Using the Distributive Property of Multiplication over Addition

Multiplying two binomials by the FOIL Method

Get Started

- Divide the students into pairs. Assign each pair of students two binomials to multiply. Make sure the factors of the binomials are the same except for the second term of the second factor. It needs to have the opposite sign from the second term of the first factor. For example, assign problems such as: $(x + 1)(x - 1)$, $(x - 2)(x + 2)$, $(x + 3)(x - 3)$, etc. Students should find the products and write answers on the board. Solutions to these problems are $x^2 - 1$, $x^2 - 4$, $x^2 - 9$, etc.
- Ask, "What do these binomial pairs have in common?" Students' response should be each binomial has the same terms but a different middle sign. Remind students this is the difference of squares.
- Ask, "What do the products have in common?" Students' response should be that the products have two terms, the second term is negative, and both terms are perfect squares. Ask leading questions, if necessary, to draw out these conclusions.
- Explain that these concepts will be used in the latter half of the lesson.

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Section 1

Expand Their Horizons

In Section 1, students will increase their understanding of radicals to include division. The Quotient Property of Square Roots and the technique of rationalizing monomial denominators are studied as methods for further simplification of radicals.

The Quotient Property of Square Roots states the square root of a quotient is equal to the quotient of the square roots: for $a \geq 0$ and $b > 0$, $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$. This property can be applied in either direction. For example, $\sqrt{\frac{9}{49}}$ can be simplified by writing it as $\frac{\sqrt{9}}{\sqrt{49}}$, which is $\frac{3}{7}$. Also, $\frac{\sqrt{10}}{\sqrt{2}}$ can be simplified by writing it as $\sqrt{\frac{10}{2}}$, which is $\sqrt{5}$.

1 Using the Quotient Property of Square Roots, the expression can be written $\frac{\sqrt{36}}{\sqrt{z^2}}$. Since $\sqrt{36}$ is six and $\sqrt{z^2}$ is $|z|$, the expression simplifies to $\frac{6}{|z|}$.

Common Error Alert

Some students may still think that $\sqrt{\frac{36}{z^2}}$ has two solutions: $\frac{6}{z}$ or $\frac{6}{-z}$. Remind students they learned every positive number has two square roots, but the radical sign is used to indicate the nonnegative or principal square root. Therefore, the expression $\sqrt{z^2}$ implies the positive root $|z|$.

2 The fraction in the problem can be written as $\sqrt{\frac{20}{5}}$. Simplify the fraction within the radical to $\sqrt{4}$. Since four is a perfect square whose square root is two, this expression simplifies to two.

Remind students that a radical expression is in simplest form when there are no perfect square (cube) factors other than one under the radical sign; there are no fractions under the radical sign; and there are no radicals in the denominator. Radicals of non-perfect squares are irrational numbers. For a radical expression to be simplified, it cannot contain a radical (irrational) number in the denominator. To rationalize the denominator is a technique whereby the radical is eliminated from the denominator, so the denominator is a rational number. The technique does not change the value of the expression.

Common Error Alert

Students commonly stop once they have reduced the fractional radicand. Remind student, in order to simplify the expression completely, there can be no perfect square factors other than one under the radical.

To rationalize a denominator, multiply it by another radical with the same index so the radicand becomes a perfect square (cube). Multiplying by the identical radical will work for square roots although there are sometimes smaller radicands that provide a shorter route to simplification. It is important to point out to students, in order to maintain the value of the fraction, whatever operation is performed on the denominator must also be performed on the numerator. In effect, the fraction is multiplied by a term over itself, which equals one; multiplication by one does not change the value of a fraction.

Consider the value $\sqrt{\frac{2}{3}}$. By using the Quotient Property of Square Roots, this can be written $\frac{\sqrt{2}}{\sqrt{3}}$. These radicals are in simplest form individually, but there is still a radical in the denominator. To eliminate this radical, first determine what radical should be multiplied in the denominator to create a perfect square radicand in the denominator. In this case, it is $\sqrt{3}$. To maintain the value of the fraction, multiply by $\frac{\sqrt{3}}{\sqrt{3}}$, a fraction whose value is one:

$$\sqrt{\frac{2}{3}} = \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{6}}{3}.$$

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An alternative approach is to determine the smallest radicand that could be multiplied with the denominator to produce a perfect square radicand. In the case $\frac{6}{\sqrt{8}}$, this would be $\sqrt{2}$. The steps are $\frac{6}{\sqrt{8}} = \frac{6}{\sqrt{8}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{6\sqrt{2}}{\sqrt{16}} = \frac{6\sqrt{2}}{4} = \frac{3\sqrt{2}}{2}$. Notice the value $\sqrt{8}$ could have been simplified first, thereby making the numbers to simplify smaller.

Common Error Alert

It is common for students to forget to multiply both the numerator and denominator by a common term when simplifying an expression. Whichever technique is chosen, be sure the numerator and denominator are multiplied by the same expression in the simplification process.

3 Multiply both the numerator and the denominator by $\sqrt{3}$ to produce a perfect square radicand in the denominator.

The denominator $\sqrt{12} \cdot \sqrt{3}$ equals $\sqrt{36}$ or six. So, $\frac{1}{\sqrt{12}}$ equals $\frac{\sqrt{3}}{6}$.

Common Error Alert

Students may try to reduce radicands with non-radicands, especially when written as a fraction. Stress to students both the numerator and the denominator must be "inside" the radical to divide or reduce as a fraction. The fraction $\sqrt{\frac{6}{2}}$ can be reduced to $\sqrt{3}$ since both terms are under the radical sign, but the fraction $\frac{\sqrt{6}}{3}$ is in simplest form.

4 Using the Quotient Property of Square Roots, the fraction can be written as $\frac{\sqrt{d^2}}{\sqrt{6}}$. Since $\sqrt{d^2} = |d|$ and six has no perfect square factors other than one, the fraction simplifies to $\frac{|d|}{\sqrt{6}}$. To eliminate the radical from the denominator, it must be rationalized: $\frac{|d|}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \frac{|d|\sqrt{6}}{6}$.

Additional Examples

1. Simplify: $\frac{6}{5\sqrt{3}}$.

To simplify, multiply the numerator and the denominator by $\sqrt{3}$.

$$\frac{6}{5\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{6\sqrt{3}}{5 \cdot 3}$$
$$= \frac{2\sqrt{3}}{5}$$

2. Simplify: $\sqrt[3]{\frac{5}{4}}$.

$$\sqrt[3]{\frac{5}{4}} = \frac{\sqrt[3]{5}}{\sqrt[3]{4}}$$
$$= \frac{\sqrt[3]{5}}{\sqrt[3]{4}} \cdot \frac{\sqrt[3]{4}}{\sqrt[3]{4}} \cdot \frac{\sqrt[3]{4}}{\sqrt[3]{4}}$$
$$= \frac{\sqrt[3]{5 \cdot 4 \cdot 4}}{4}$$
$$= \frac{2\sqrt[3]{10}}{4}$$
$$= \frac{\sqrt[3]{10}}{2}$$

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Section 2

Expand Their Horizons

In Section 2, students will broaden their knowledge of rationalizing the denominator to include binomial denominators with radical expressions. To simplify these expressions, the denominator must be multiplied by its conjugate. Recall the difference of squares formula: $a^2 - b^2 = (a + b)(a - b)$. The factor containing the addition symbol is conjugate to the factor with the subtraction symbol. When multiplied together, the first and last terms were squared, and the middle terms canceled, thus eliminating the radical expressions in the binomial.

For example, the binomial $2 + \sqrt{3}$ is conjugate to $2 - \sqrt{3}$. The product of these two expressions is $(2 + \sqrt{3})(2 - \sqrt{3}) = 2 \cdot 2 - 2\sqrt{3} + 2\sqrt{3} - \sqrt{3} \cdot \sqrt{3} = 2^2 - \sqrt{3}^2 = 4 - 3 = 1$.

Connections

Obtaining accurate decimal approximations of ratios containing radicals in their denominator was difficult before the invention of personal calculators. The decimal representation of an irrational number is a non-repeating, non-terminating decimal. Imagine long division with a divisor like that! Take for example $\frac{\sqrt{5}}{\sqrt{2}}$.

Using decimal approximations of each, the calculation would be $2.236067978 \div 1.414213562$. However, if the denominator is rationalized, the fraction becomes $\frac{\sqrt{10}}{2}$, and the division with a decimal approximation is $3.16227766 \div 2$, which is much easier to calculate. Even today, with prevalent calculator usage, computations are much easier with a rationalized denominator.

If the technique of conjugates is used to rationalize a binomial expression containing radicals, remember to multiply the conjugate factor in both the numerator and the denominator.

5 The conjugate of $1 + \sqrt{3}$ is $1 - \sqrt{3}$. Use the FOIL Method to simplify the denominator.

$$\frac{1}{1 + \sqrt{3}} = \frac{1}{1 + \sqrt{3}} \cdot \frac{1 - \sqrt{3}}{1 - \sqrt{3}} =$$

$$\frac{1 - \sqrt{3}}{1 + \sqrt{3} - \sqrt{3} - 3} = \frac{1 - \sqrt{3}}{-2}.$$

Distributing in the numerator yields $1 - \sqrt{3}^2$, while multiplication by the conjugate in the denominator yields $1^2 - \sqrt{3}^2$, which is $1 - 3$ or -2. The simplified fraction is $\frac{1 - \sqrt{3}}{-2}$. Some people prefer not to have negative numbers in the denominator. In that case, $\frac{1 - \sqrt{3}}{-2}$ would be multiplied by $\frac{-1}{-1}$, in effect, changing all signs throughout the fraction: $\frac{-1 + \sqrt{3}}{2}$.

Look Beyond

Trigonometry and calculus use the rationalization of denominators to simplify fractions containing radicals. The calculation of trigonometric functions often requires exact values rather than decimal approximations. In particular, sine and cosine functions with an angle measurement of 45°, as well as those for 30° and 60° angles, are commonly used in their radical form.

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Additional Examples

1. Simplify: $\dfrac{5\sqrt{6}}{3\sqrt{2} - 2\sqrt{3}}$.

Multiply both the numerator and the denominator by the denominator's conjugate. Then, use the difference of squares and simplify.

$$\frac{5\sqrt{6}}{3\sqrt{2} - 2\sqrt{3}} = \frac{5\sqrt{6}}{3\sqrt{2} - 2\sqrt{3}} \cdot \frac{3\sqrt{2} + 2\sqrt{3}}{3\sqrt{2} + 2\sqrt{3}}$$

$$= \frac{15\sqrt{12} + 10\sqrt{18}}{(3\sqrt{2})^2 - (2\sqrt{3})^2}$$

$$= \frac{30\sqrt{3} + 30\sqrt{2}}{9 \cdot 2 - 4 \cdot 3}$$

$$= \frac{30(\sqrt{3} + 30\sqrt{2})}{6}$$

$$= \frac{30(\sqrt{3} + \sqrt{2})}{6}$$

$$= 5(\sqrt{3} + \sqrt{2})$$

2. Simplify: $\dfrac{5 - \sqrt{2}}{2 + \sqrt{3}}$.

Rationalize the denominator by multiplying both the numerator and the denominator by its conjugate $2 - \sqrt{3}$ and then, simplify.

$$\frac{5 - \sqrt{2}}{2 + \sqrt{3}} = \frac{5 - \sqrt{2}}{2 + \sqrt{3}} \cdot \frac{2 - \sqrt{3}}{2 - \sqrt{3}}$$

$$= \frac{5 \cdot 2 - 5\sqrt{3} - 2\sqrt{2} + \sqrt{2} \cdot \sqrt{3}}{(2)^2 - (\sqrt{3})^2}$$

$$= \frac{10 - 5\sqrt{3} - 2\sqrt{2} + \sqrt{6}}{4 - 3}$$

$$= 10 - 5\sqrt{3} - 2\sqrt{2} + \sqrt{6}$$

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