17.1 teacher notes

Objectives

- Simplify square root expressions.
- Simplify cube root expressions.

Vocabulary

Square root (Lesson 1-4)
Radical sign (Lesson 1-4)
Radicand (Lesson 1-4)
Product Property of Square Roots
Simplest form of a radical
Perfect square
Cube
Cube root
Perfect cube
Index
Product Property of Cube Roots

Prerequisites

Simplifying square (cube) roots of perfect squares (cubes)

Finding the square (cube) of positive and negative values

Finding absolute values

Get Started

- Remind students the area of a rectangle is found by multiplying length by width. Draw a square with a side length of five units. Ask students for its area. The correct response is 25 square units.
- Divide students into pairs. Ask each pair to create at least three different squares using whole number lengths and find their areas.
- List the areas found by some student pairs. Ask students what these numbers have in common, leading them to conclude they are perfect squares.
- Remind students the volume of a box is found by multiplying length by width by height. Draw a cube with edge lengths of five units. Ask students for its volume. The correct response is 125 cubic units.
- Ask each student pair to think of at least three different cubes using whole number lengths and find their volumes.
- List the volumes found by some student pairs. Ask students what these numbers have in common, leading them to conclude the numbers are perfect cubes. Explain numbers to the third power are called cubes because the volume of a cube is found in this manner.

Section 1

Expand Their Horizons

In Section 1, students will expand their understanding beyond simplifying perfect squares to simplifying numbers which contain perfect square factors. Negative and positive roots and the roots of negative and positive numbers are examined.

A radical without an index is understood to have an index of two. The symbol $\sqrt{}$ is often referred to as the square root symbol. The square root of a number is the number whose square is that number: $a = \sqrt{b}$ implies $a^2 = b$. For example, $\sqrt{25}$ is 5 because $5^2 = 25$. Since a positive number squared is positive, as with $5^2 = 25$, and a negative number squared is positive, as with $(-5)^2 = 25$, a number actually has two square roots, one positive and one negative. The radical symbol indicates the principal square root, which is nonnegative. The negative root is indicated with a negative before the radical, as $-\sqrt{25} = -5$.

The square root of a whole number is simplified when the radicand (the number under the radical) contains no perfect square factors other than the number one, and there is no radical in the denominator of a fraction. To simplify or reduce a radical, list all factors of the radicand and choose a factor pair that contains a perfect square. Consider the value $\sqrt{50}$; its factors are 1, 2, 5, 10, 25, and 50. Of these factors, 25 is a perfect square. Choose the pair of factors 2 and 25 to equal 50. The Product Property of Square Roots states the square root of the product is equal to the product of the square roots. Write the radical expression as the product of its chosen factors. For $\sqrt{50}$, the product would be $\sqrt{2} \cdot \sqrt{25}$. Take the square root of the perfect square and leave the other factor in its radical form. In this example, $\sqrt{50} = \sqrt{2} \cdot \sqrt{25} = \sqrt{2} \cdot 5 = 5\sqrt{2}$. The value 5 placed on the left side of the radical so as to not confuse 5 as being under the radical symbol. This can be read this answer as either "five times the square root of two" or "five root two."

Using the largest perfect square factor of the radicand will shorten the number of steps needed to simplify the radical. For example, the largest perfect square factor of 72 is 36; this gives $\sqrt{72} = \sqrt{36}\sqrt{2} = 6\sqrt{2}$. This method is equivalent to $\sqrt{72} = \sqrt{9}\sqrt{8} = 3\sqrt{8} = 3\sqrt{4}\sqrt{2} = 3 \cdot 2\sqrt{2} = 6\sqrt{2}$ and $\sqrt{72} = \sqrt{4}\sqrt{18} = 2\sqrt{18} = 2\sqrt{9}\sqrt{2} = 2 \cdot 3\sqrt{2} = 6\sqrt{2}$.

Common Error Alert

Students may have trouble remembering which factor is brought out of the radical. Remind students, if two factors of the radicand are identical, then this factor is a perfect square, and it may be factored out of the radical. For example, $\sqrt{108} = \sqrt{6 \cdot 6 \cdot 3} = 6\sqrt{3}$.

An alternate method for simplifying the radical expression would be to use the prime factorization of the radicand. A perfect square is a number created by multiplying two of the same number. Again, consider $\sqrt{50}$. Its prime factorization is $2 \cdot 5 \cdot 5$. Since there is a pair of 5's, one five is placed outside the radical, and the two remains inside the radical. So, $\sqrt{50} = 5\sqrt{2}$. This method may be helpful to students having trouble remembering perfect squares. The value $\sqrt{72}$ has the prime factorization of $2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$; one of the pair of 2's and one of the pair of 3's is placed outside the radical and remaining factor 2 is left within the radical. So, $\sqrt{72} = 2 \cdot 3\sqrt{2} = 6\sqrt{2}$.

Common Error Alert

Students may mistakenly believe the square root of a sum may be found by applying a property similar to the Product Property of Square Roots. However, $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$. To illustrate this, demonstrate finding the square root of the sum yields a different result than finding the sum of the square of the terms. The value $\sqrt{9+16} = \sqrt{25} = 5$, but $\sqrt{9} + \sqrt{16} = 3 + 4 = 7$, and five is not equal to seven.

Negative numbers do not have real number square roots. Because the square of a real number is either zero or a positive number, a negative number cannot be the square of another real number. A positive number has two square roots with the principle square root being positive. In higher mathematics, students will learn the square root of a negative number is called an imaginary or complex number.

1 The factors of 63 are 1, 3, 7, 9, 21, and 63. Since nine is a perfect square factor, write $\sqrt{63}$ as $\sqrt{9 \cdot 7} = \sqrt{9}\sqrt{7}$ using the Product Property of Square Roots. The square root of the perfect square nine is three, and this number is placed outside the radical leaving $3\sqrt{7}$. Because seven has no perfect square factor other than one, this is in simplest form.

2 The factors of 80 are 1, 2, 4, 5, 8, 10, 16, 20, 40, and 80. Both 4 and 16 are perfect squares. Choosing the larger perfect square and using the Product Property of Square Roots, $\sqrt{80}$ may be written as $\sqrt{16}\sqrt{5}$ which equals $4\sqrt{5}$. The square root of the perfect square is placed outside the radical, and because 5 has no perfect square factors other than one, this is in simplest form. If the factor pair 4 and 20 was used, the radical would need to be simplified twice: $\sqrt{80} = \sqrt{4}\sqrt{20} = 2\sqrt{20} = 2\sqrt{4}\sqrt{5} = 2 \cdot 2\sqrt{5} = 4\sqrt{5}$. The result, however, is the same.

3 The square root of any negative number is not a real number; $\sqrt{-8}$ is not a real number. For the sake of argument, say $x = \sqrt{-8}$; then, $x^2 = -8$, but the square of any real number is nonnegative. So, $\sqrt{-8}$ is not a real number.

Additional Examples

1. Compare: $\sqrt{-9}$ and $-\sqrt{9}$.

The square root of a negative number is not a real number; thus, $\sqrt{-9}$ is not a real number. There is no real number that, when squared, yields a negative result. However, a negative sign before a radical indicates the opposite of the principle root. Because $\sqrt{9}$ is 3, then $-\sqrt{9}$ is -3.

2. Simplify: $\sqrt{b^3}$.

The factors of b^3 are b, b, and b. Therefore, $\sqrt{b^3} = \sqrt{b \cdot b \cdot b}$. This can be simplified by removing one pair of b's from the radical and rewriting the expression as $b\sqrt{b}$.

Section 2

Expand Their Horizons

In Section 2, students will learn to calculate cube roots of perfect cubes and to simplify other cubes by using factors which contain perfect cubes. Cube roots of negative and positive numbers are examined.

The index of the radical indicates the root. An index of 3 indicates a 3rd root, which is often referred to as a cube root; an index of 4 indicates a 4th root, etc. In the expression $\sqrt[n]{a}$, the index is n and represents the nth root of the radicand a. A cube root is a number whose cube is the radicand. For example, $\sqrt[3]{125}$ is 5 because $5^3 = 125$. Because some students find perfect cubes more difficult to recognize than perfect squares, it may be helpful to cube several numbers before beginning this section: $2^3 = 8$, $3^3 = 27$, $4^3 = 64$, etc.

Unlike square roots, negative numbers do have real number cube roots. Because the cube of a positive number is a positive number and the cube of a negative number is a negative number, a positive number has one positive cube root, and a negative number has one negative cube root. For radicals with an even index (like square roots, 4th roots, and 6th roots), the root of a positive number can be positive or negative, but the root of a negative number is not a real number. For a radical with an odd index (like cube roots, 5th roots, and 7th roots), the root of a positive number is only positive, and the root of a negative number is only negative.

A cube root of a whole number is in simplest form when the radicand contains no perfect cube factors other than the number one. Like square roots, to simplify a cube root, list all factors of the radicand and choose a pair of factors that contains a perfect cube. The Product Property of Cube Roots states that the cube root of the product is equal to the product of the cube roots: $\sqrt[3]{ab} = \sqrt[3]{a}\sqrt[3]{b}$. Consider $\sqrt[3]{40}$. The radicand's factors are 1, 2, 4, 5, 8, 10, 20, and 40. Since eight is a perfect cube, this can be rewritten as $\sqrt[3]{8}\sqrt[3]{5}$ which is equal to $2\sqrt[3]{5}$. Again, using the largest perfect cube will shorten the number of steps needed to find the simplified radical, and the radicand should be reduced until it has no perfect cube factors other than the number one.

Common Error Alert

Students may inadvertently omit the index from the reduced radical. Because a radical sign written without an index is understood to be a square root, this changes the root and is, therefore, incorrect. Stress that the index is a part of the radical sign and should remain the same as the original problem in every step of the simplification process.

Prime factorization may be used as an alternate method for simplifying cube roots. A perfect cube is created by multiplying three of the same number together. Therefore, when simplifying using primes, one of any trio of duplicate factors will be moved outside the radical. Consider the example $\sqrt[3]{40}$ again. The number 40 has a prime factorization $2 \cdot 2 \cdot 2 \cdot 5$. Since there is a trio of 2's, they are removed from inside the radical, and one 2 is placed outside the radical. The 5 remains inside the radical because it cannot be reduced further. Thus, $\sqrt[3]{40} = 2\sqrt[3]{5}$. Two or more factors as coefficients or as radicands should be multiplied together. For example, $\sqrt[3]{108}$ has the prime factorization of $2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$; one of the trio of 3's is placed outside the radical and remaining factors are left within the radical. Thus, $\sqrt[3]{108} = 3\sqrt[3]{2 \cdot 2} = 3\sqrt[3]{4}$.

Connections

Because finding an accurate approximation of a non-perfect square number was sometimes difficult and time consuming before calculators, tables of radical values were often used as approximations. To minimize the size of these tables, numbers with perfect square factors were not included. If a student needed, for example $\sqrt{12}$, he or she would rewrite it as $2\sqrt{3}$. Then, the student would find the approximate value of $\sqrt{3}$ in the table and multiply the result by the coefficient 2.

Radical expressions are more accurate than decimal approximations in the same way that the fraction $\frac{1}{3}$ is more accurate than its decimal approximation 0.33. Additionally, the manipulations used to write a radical in a different but equivalent form strengthens one's skills for future mathematics. The use of radical equations is used extensively in higher mathematics and physics. The ability to manipulate these expressions may reduce many steps in a computational process.

4 The factors of 128 are 1, 2, 4, 8, 16, 32, 64, and 128. Both 8 and 64 are perfect cubes. Choosing the larger perfect cube and using the Product Property of Cube Roots, $\sqrt[3]{128}$ can be written as $\sqrt[3]{64} \cdot \sqrt[3]{2}$, which equals $4\sqrt[3]{2}$. The cube root of the perfect cube is placed outside the radical, and because 2 has no perfect cube factors other than one, this is simplest form. If the factor pair 8 and 16 was used, the radical would need to be simplified two times; the result, however, would be the same ($\sqrt[3]{128} = \sqrt[3]{8}\sqrt[3]{16} = 2\sqrt[3]{16} = 2\sqrt[3]{8}\sqrt[3]{2} = 2 \cdot 2\sqrt[3]{2} = 4\sqrt[3]{2}$).

5 Factors of 500 are 1, 2, 4, 5, 10, 20, 25, 50, 100, 125, 250, and 500. Because the radicand is negative and the cube root of a negative number is a negative number, choose the perfect cube 125 written as a negative factor. Thus, $\sqrt[3]{-500}$ can be written as $\sqrt[3]{-125}\sqrt[3]{4}$, which simplifies to $-5\sqrt[3]{4}$. The radicand 4 has no perfect cube factors other than 1, so this is in simplest form. If the value $\sqrt[3]{-500}$ is written as $5\sqrt[3]{-4}$, then the expression is not in simplest form, because the number -1 is also a perfect cube.

Not only can numbers be under a radical symbol, but variables can also as well. In the expression $\sqrt{(-7)^2}$, notice this equals $\sqrt{|-7|} \cdot \sqrt{|-7|}$. The absolute values are inserted because the square of any real number is nonnegative. The expression $\sqrt{|-7|} \cdot \sqrt{|-7|}$ equals $|-7|$. Similarly, let x equal some number. Then, the expression is $\sqrt{x^2} = \sqrt{|x|} \cdot \sqrt{|x|}$ or $|x|$. Here the absolute value is needed because it is not known whether or not x is negative. For cube roots, the cube root of a number cubed is that number. For example, $\sqrt[3]{5^3} = \sqrt[3]{5} \cdot \sqrt[3]{5} \cdot \sqrt[3]{5} = 5$; or $\sqrt[3]{-5^3} = \sqrt[3]{-5} \cdot \sqrt[3]{-5} \cdot \sqrt[3]{-5} = -5$. So, $\sqrt[3]{x^3} = x$ for all real numbers. Students will learn more about radical expressions containing variables in future lessons.

Look Beyond

Reduced radicals are used to ease computations in which the radical form of an expression is preferred over the decimal approximation of the root. Geometry and trigonometry make great use of these techniques, particularly in the application of the Pythagorean Theorem. A consequence of the Pythagorean Theorem is the distance formula. In the coordinate plane, the distance, D, between two points (a, b) and (c, d) is given by the formula $D = \sqrt{(c - a)^2 + (d - b)^2}$.

Additional Examples

1. Compare: $\sqrt[3]{-64}$ and $-\sqrt[3]{64}$.

The simplified form of both expressions is -4 but for different reasons. The expression $\sqrt[3]{-64}$ can be written as $\sqrt[3]{(-4)^3}$ which equals -4. The expression $-\sqrt[3]{64}$ indicates the opposite of $\sqrt[3]{64}$. Since $\sqrt[3]{64}$ is 4, its opposite is -4.

2. Simplify: $\sqrt[3]{4} \cdot \sqrt[3]{12}$.

By the Product Property of Cube Roots, $\sqrt[3]{4} \cdot \sqrt[3]{12}$ can be written as $\sqrt[3]{4 \cdot 12}$ or $\sqrt[3]{48}$. This equals $\sqrt[3]{8 \cdot 6}$ or $\sqrt[3]{8} \cdot \sqrt[3]{6}$. Since $\sqrt[3]{8} = 2$, the expression may be simplified to $2\sqrt[3]{6}$.