16.2 teacher notes

Objectives

- Determine whether a function is a direct variation and identify the constant of variation.
- Solve problems using direct variation.

Vocabulary

Function (Lesson 9-1)
Direct variation
Ratio
Cross multiplication
Constant (Lesson 2-1)
Quotient (Lesson 1-2)
Constant of variation
Ordered pair (Lesson 7-1)
Proportion

Prerequisites

Reducing fractions

Solving linear equations

Get Started

- Say, "One cup equals eight ounces. If we have two cups, how many ounces are there?" Students should reply 16 ounces.
- Repeat the question for three and four cups. With each response, write a corresponding ratio such as $\frac{3 \text{ cups}}{24 \text{ ounces}} = \frac{1}{8}$ cup per ounce.
- Instruct students to create their own scenarios using units such as dollars per pound, inches per feet, or miles per hour.
- Finally, introduce a variable x. Say, "If a car is cruising at 55 miles per hour for three hours, how many miles, x, has the car gone?" It may help by setting up the proportion $\frac{55 \text{ mi}}{1 \text{ h}} = \frac{x \text{ mi}}{3 \text{ h}}$. The correct response is 165 miles.
- Tell students these relationships are examples of a functional relationship called direct variation, which will be studied in this lesson.

Section 1

Expand Their Horizons

In Section 1, students will determine whether a function represents a direct variation by creating ratios for corresponding y and x values. If this nonzero ratio is the same for all xy pairs, the function represents a direct variation. This quotient $\frac{y}{x}$ is called the constant of variation that can be used to find other values for the function.

Given a table of values for a function, ratios of y to x should be created for each ordered pair. If the ratios for every xy pair are equal and nonzero, then the function is a direct variation; we say "y varies directly as x."

1 The ratio of y to x is found for each ordered pair. The first ratio is $\frac{6}{8}$, which reduces to $\frac{3}{4}$. The second ratio is $\frac{9}{12}$, which reduces to $\frac{3}{4}$. The last ratio $\frac{10}{15}$ reduces to $\frac{2}{3}$. Because the ratios are not the same, this is not a direct variation function.

2 The ratio of y to x is found for each ordered pair. The first ratio is $\frac{15}{3} = 5$. The second ratio is $\frac{25}{5} = 5$. The last ratio is $\frac{-10}{-2} = 5$. The ratio is the same for each pair, so this table represents a direct variation.

3 The previous problem determined the table of values represented a direct variation; each ratio reduced to 5, which is the constant of variation. An equation for the direct variation would be $\frac{y}{x} = 5$, because each ratio of each y value to each x value was the constant 5.

The general form of a direct variation equation is $\frac{y}{x} = k$, with k being the constant of variation. This equation can also be written in the form $y = kx$. In this latter case, k is sometimes referred to as the conversion factor.

Implicit in the assumption the table of values in Problems 2 and 3 represents a direct variation is the notion y *always* varies directly as x. In the table of values, we determine that y varies directly as x for the given values. In forming the equation $\frac{y}{x} = 5$, the assumption is made that y varies directly as x for **all** values of x (except the point where $x = 0$ because $\frac{y}{0}$ is undefined).

Additional Examples

1. Find the values of y for the function below if y varies directly as x, and the constant of variation, k, is 3.

x	y
4	
7	
–5	

Solve as $y = kx$
$y = 3(4) = 12$
$y = 3(7) = 21$
$y = 3(-5) = -15$

2. The table represents a direct variation function with a constant of variation $k = 2$. Fill in the table with its missing values.

x	y
2	
	10
–1	

Solve as $\frac{y}{x} = k = 2$
$\frac{y}{2} = 2;\ y = 2 \cdot 2 = 4$
$\frac{10}{x} = 2;\ 10 = 2x;\ x = 5$
$\frac{y}{-1} = 2;\ y = 2 \cdot -1 = -2$

Section 2

Expand Their Horizons

In Section 2, students will use Section 1's concept of direct variation to solve application problems. Cross multiplication is introduced as a method used for solving these problems.

A proportion is a statement of equality between two ratios or fractions. Cross multiplication is used to rewrite the proportion $\frac{a}{b} = \frac{c}{d}$, where $b \neq 0$ and $d \neq 0$, in the form $ad = bc$. The values ad and bc are called cross products.

In the fractional equation above, a common denominator is bd. Multiplying both sides of the equation by this common factor gives $bd\left(\frac{a}{b}\right) = bd\left(\frac{c}{d}\right)$. The b's cancel on the left side of the equation, and the d's cancel on the right side leaving $da = bc$, which is equivalent to $ad = bc$ according to the Commutative Property of Multiplication. Another method of explaining cross multiplication is based upon the necessity for a common denominator when adding or subtracting fractions. Rewrite the equation as $\frac{a}{b} - \frac{c}{d} = 0$. Find a common denominator to subtract the fractions; again, bd is a common denominator. Writing this equation with its common denominator and combining terms gives $\frac{ad - bc}{bd} = 0$. The implication which is drawn because a fraction is equal to zero only when the numerator is equal to zero is $ad - cb = 0$. By using the Addition Property of Equality, the equation given by cross multiplication is $ad = bc$.

Common Error Alert

Students should understand cross multiplication is only used in solving proportions. Equations with more than one fraction on either side of the equal sign must be solved using an alternative method.

4 A map is a scaled representation of actual distance. For this reason, map measurements vary directly as actual distance; their ratios are constant. Given five inches on a map corresponds to 65 miles in actual distance, the ratio is $\frac{y}{x} = \frac{5}{65}$, where x is actual distance and y is map distance. The constant of direct variation $\frac{5}{65}$ can be reduced to $\frac{1}{13}$. Then, the proportion is $\frac{1}{13} = \frac{3}{m}$; cross multiplication of this equation also gives $m = 39$ miles.The form $y = kx$ may be used to solve the problem. With $k = \frac{1}{13}$ and $y = 3$, the corresponding actual distance x can be found: $3 = \frac{1}{13}x$ which again yields $x = 39$. Multiplication by the ratio $\frac{1}{13}$ converts an actual distance into a map distance, which is why k is called a conversion factor in this form of the equation.

5 The distance formula $d = rt$ can be rewritten in the form $\frac{t}{d} = \frac{1}{r}$. Because the speed of sound, r, through air is constant, $\frac{1}{r} = k$ is the constant of direct variation. Time is y and distance is x. By cross multiplication, the equation $\frac{10}{2} = \frac{4}{d}$ can be written in an equivalent form $10d = (4)(2)$.

Common Error Alert

It is important students be consistent in the placement of terms in a proportion. For a direct variation problem, one method is to write the proportion so that both numerators represent the same unit of measure and both denominators represent the same unit of measure. A second method is using the same units on either side of the equation. For example, a problem asking how many ounces are in four cups can be solved with either of these proportions.

$$\frac{1 \text{ cup}}{8 \text{ ounces}} = \frac{4 \text{ cups}}{x \text{ ounces}} \qquad \frac{1 \text{ cup}}{4 \text{ cups}} = \frac{8 \text{ ounces}}{x \text{ ounces}}$$

In some applications, direct variation may be implied rather than stated explicitly. When an increase in one quantity causes a proportional increase in the other quantity, a direct variation or direct proportion exists.

For example, the relationship between the distance traveled on a trip and the gallons of gasoline used may be a direct proportion; the greater the distance traveled the more gasoline is used. However, the relationship between the speed driven on the trip and the total time of the trip is an inverse proportion; the greater the speed driven, the less time it takes to complete the trip. Inverse variation will be the subject of Lesson 16-3.

Connections

Direct variation is one of the most useful mathematical concepts for day-to-day life. Suppose a recipe requires two pounds of eggplant for eight servings, and there are 12 dinner guests; the direct variation function can be used to determine how much eggplant must be purchased. Suppose the height of a tree must be determined before it is cut down; the length of its shadows varies directly with the height of an object, so the height and shadow length of a smaller object can be used with the shadow of the tree to find its height.

Look Beyond

In geometry, students will study other applications of direct variation. One example of direct variation in geometry is finding corresponding side lengths of similar triangles. Similar triangles are triangles that have the same shape but are different sizes; the lengths of the sides of such triangles are proportional (vary directly). Lengths are found with direct variation equations and cross multiplication.

Additional Examples

1. The gallons of gasoline used for a trip varies directly with the distance traveled. If a driver uses 15 gallons of gasoline for a 360-mile trip, how many gallons will be used for a trip that is 600 miles long?

$$\frac{15 \text{ gallons}}{x \text{ gallons}} = \frac{360 \text{ miles}}{600 \text{ miles}}$$

$$\frac{15}{x} = \frac{360}{600}$$

$$9{,}000 = 360x$$

$$25 = x$$

So, 25 gallons of gasoline are used for a 600-mile trip.

2. A salesman earns $100 for every $2,000 of product he sells. If he sells $7,500 of product, how much will he earn?

$$\frac{100}{y} = \frac{2{,}000}{7{,}500}$$

$$\frac{100}{y} = \frac{2{,}000}{7{,}500}$$

$$750{,}000 = 2{,}000y$$

$$375 = y$$

So, he earns $375 for $7,500 of product sales.