16.1 teacher notes

Objectives

- Solve rational equations by multiplying the equation by the least common multiple of the denominators.
- Identify when a number is a restricted value of a variable and recognize that it is not a solution.

Vocabulary

Rational equation
Least common denominator (Lesson 1-3)
Least common multiple (Lesson 1-3)
Distributive Property of Multiplication over Addition (Lesson 2-3)
Factor (Lesson 1-3)
Restricted value (Lesson 15-1)

Prerequisites

Finding the least common multiple

Multiplying and dividing with fractions

Adding and subtracting with fractions

Applying the distributive property to remove parenthesis

Get Started

- Break the class into pairs. Give students the equation $\frac{5}{6}x - \frac{1}{4} = \frac{1}{4}x + \frac{3}{2}$. Tell students to solve this equation by moving the variables to one side of the equation and the constants to the other side. As students are working, remind them addition and subtraction require a least common denominator and division involves multiplying by the reciprocal of the divisor.
- Give students the second equation $10x - 3 = 3x + 18$ to solve.
- Say, “Which equation was easier to solve?” Allow discussion. Students will note that the fractions made the first equation much harder to solve.
- Explain to the students in this lesson, they will learn to eliminate fractions in an equation in the first step. Show them that by multiplying the first equation by a common denominator of 12 will result in the second equation they solved.

Section 1

Expand Their Horizons

In Section 1, students review the concepts of rational expressions and least common denominators. They will apply these concepts to solve rational equations by first eliminating the fractions. The steps used for solving rational equations are

(1) determining the LCD,
(2) multiplying each side of the equation by the LCD,
(3) solving the resulting equation, and
(4) checking the answer to see if it is a restricted value.

Because students are now experienced with solving equations by using properties of equality to both sides of an equation, solve a simple proportion using this approach. For example, in the equation $\frac{x}{6} = \frac{4}{3}$ the Multiplication Property of Equality is used to isolate the variable x on the left hand side of the equation. This gives $6\left(\frac{x}{6}\right) = 6\left(\frac{4}{3}\right)$, so $x = 8$.

Next, solve a rational proportion with a variable on each side and demonstrate that the variable is divided by two distinct numbers and must, therefore, be multiplied by these individual numbers. For example, in the equation $\frac{x+1}{4} = \frac{x+2}{6}$, use the Multiplication Property of Equality to multiply each side of the equation by four and by six. This is equivalent to multiplying each side by 24. This gives $24\left(\frac{x+1}{4}\right) = 24\left(\frac{x+2}{6}\right)$, which simplifies to $6x + 6 = 4x + 8$, so $x = 1$. Students may wonder why cross multiplication was not used to solve this equation. Let students know that cross multiplication is only *one* possible method for solving proportions. In this lesson, however, students will be able to change *all* rational equations into equations containing no denominators.

Finally, demonstrate for the students that multiplication by 12, the least common denominator, would give the same result while at the same time give the advantage of working with smaller numbers. In the previous example, this would give $12\left(\frac{x+1}{4}\right) = 12\left(\frac{x+2}{6}\right)$, which simplifies to $3x + 3 = 2x + 4$, and the solution is again x equals one. This method is referred to as "eliminating fractions" in this lesson; it may also be referred to as "clearing fractions" or "clearing the denominators."

Common Error Alert

Students may attempt to cross multiply on all rational equations. Cross multiplication is a method used for solving proportions. If there is only one fraction on each side of the equation, cross multiplication may be used; otherwise, cross multiplication is not an appropriate method.

Point out to students any common denominator will eliminate the fractions and produce the correct solution (in this example 24 resulted in the correct solution, as would 36 or 48), and multiplying the denominators together will always produce a common denominator. However, multiplying the denominators together will not always produce the least common denominator, which is the most economical choice.

1 If there is only one denominator in an equation, the LCD is that number or expression which, in this case, is 3x. On the left hand side of the equation, the LCD cancels with the given denominator, leaving two. On the right hand side of the equation, the product of 3x and 6 is 18x. Dividing both sides of the equation by the coefficient 18 yields $\frac{2}{18} = x$ or $x = \frac{1}{9}$ when the fraction is reduced. When checking the answer, note that the fraction bar functions as a grouping symbol, as well as a division symbol. Multiplication in the denominator must be done first according to the order of operations. Now, use the fact that division is the same as multiplying the numerator of the fraction by the inverse of the denominator. The answer found is a solution to the equation.

Common Error Alert

Students should understand the fraction bar indicates a grouping of the terms in denominator. For example, given the equation $\frac{3}{x+2} = \frac{1}{2}$, students may incorrectly believe that $x + 2$ is the LCD, but $(x + 2)$ and 2 are each factors of the LCD, which is $2(x + 2)$.

2 The LCD of 8, 6, and 12 is 24. This can best be found by finding the prime factorization of each number. The prime factorizations are $8 = 2 \cdot 2 \cdot 2$, $6 = 2 \cdot 3$, and $12 = 2 \cdot 2 \cdot 3$. The LCD is found by using all distinct factors and using each factor the greatest number of times it appears in any one factorization. The product of these numbers is the LCD. The prime factor 2 appears at most three times and the prime factor 3 appears at most one time. $2 \cdot 2 \cdot 2 \cdot 3 = 24$. Multiply each side of the equation by 24. This gives $3x - 4 = 2$, so $x = 2$.

The method of factorization to find the LCD can even be used for expression containing variables. For example, the LCD of $4x^2$ and $6xy$ is $12x^2y$. $4x^2 = 2 \cdot 2 \cdot x \cdot x$ and $6xy = 2 \cdot 3 \cdot x \cdot y$. The prime factor 2 appears at most two times; the prime factor 3 appears at most one time; the factor x appears at most two times; and the factor y appears at most one time. $2 \cdot 2 \cdot 3 \cdot x \cdot x \cdot y = 12x^2y$. We do not know if the variables are prime; the method shown is the only way to factor these values.

3 The LCD for the denominators 3, x, and $3x$ is $3x$. To eliminate the fractions, use the Multiplication Property of Equality and multiply $3x$ by each side of the equation. This gives the equation $x + 6 = 4$, so $x = -2$. Substitute the value -2 into x of the original equation to check the answer.

Checking an answer is always advisable. However, rational equations *must* be checked when a denominator contains a variable, because an answer might be a restricted value; in which case, a linear equation would have no solution. A restricted value results when a check of the answer reveals that a denominator is a value of zero. Division by zero is undefined, so the answer is not a solution. Obtaining an answer that is a restricted value does not indicate the equation was solved incorrectly. Answers to rational equations should be checked to assure the answer does not produce a denominator of zero.

Connections

Applied problems involving the amount of work to be done, involving the rate at which the work is done, and involving the time it takes to complete the work can be solved using the methods learned in this lesson. Rate is expressed as a ratio or fraction; for example, a person who can complete three homework problems in 10 minutes has a rate of $\frac{3}{10}$ problem per minute. Equations can be created to find the time it takes several people or machines to complete a given job working at varying rates, and the method of eliminating fractions would make the solution of that equation simpler.

Restricted values may be found before solving the equation. Set the variable expression in each denominator equal to zero and solve for the variable. For example, given an equation with a denominator of $x - 3$, the restricted value of three would be found by solving $x - 3 = 0$. Once the linear equation is solved, if three is the answer found, the equation has no solution.

Look Beyond

In Lesson 6-5, range is described as the set of possible values or possible output values of an expression. Likewise, domain is the set of possible values the variable can have in an expression. These values are not in the domain. Many equations, including those with fractions, radicals, and logarithms, have restricted values. In more advanced mathematics courses, these restrictions or domains will be found and will be used to solve and graph various functions and relations.

Additional Examples

1. Solve: $\frac{x-4}{2} - \frac{x-6}{4} = 2.$

The LCD is 4. Multiply each side by 4.

$$4\left(\frac{x-4}{2} - \frac{x-6}{4}\right) = 4(2)$$
$$4\left(\frac{x-4}{2}\right) - 4\left(\frac{x-6}{4}\right) = 4(2)$$
$$2(x-4) - 1(x-6) = 4(2)$$
$$2x - 8 - x + 6 = 8$$
$$x - 2 = 8$$
$$x = 10$$

A check will determine this is the solution.

2. Solve: $\frac{x}{x+2} + \frac{3}{2} = \frac{4x+7}{2x+4}.$

The denominator $2x + 4 = 2(x + 2)$; this is the LCD.

$$2(x+2)\left(\frac{x}{x+2} + \frac{3}{2}\right) = 2(x+2)\left(\frac{4x+7}{2x+4}\right)$$
$$2(x) + (x+2)(3) = 4x + 7$$
$$2x + 3x + 6 = 4x + 7$$
$$5x + 6 = 4x + 7$$
$$x = 1$$

A check will determine this is the solution.

Challenge

In the Challenge section of this lesson's Independent Practice, students will solve problems where, after clearing the denominator, the result is a non-linear equation. In this lesson, after they multiply both sides of the equation by the least common denominator and simplify the expression, they are left with a linear equation. Not all rational equations will simplify into a linear equation by using the methods learned in this lesson. After eliminating the fractions, they are sometimes left with a quadratic equation. For example, in the equation $\frac{x}{x+2} + \frac{2}{x-2} = \frac{x+6}{x^2-4}$, the LCD is $(x + 2)(x - 2)$. Multiply both sides of the equation to eliminate the fractions.

$$\frac{x}{x+2} + \frac{2}{x-2} = \frac{x+6}{x^2-4}$$

$(x+2)(x-2)\left(\frac{x}{x+2} + \frac{2}{x-2}\right) = (x+2)(x-2)\left(\frac{x+6}{x^2-4}\right)$	Multiply by the LCD
$x(x-2) + 2(x+2) = x + 6$	Eliminate the fractions
$x^2 - 2x + 2x + 4 = x + 6$	Distributive Property
$x^2 - x - 2 = 0$	Addition and subtraction
$(x-2)(x+1) = 0$	Factor

This gives $x = 2$ or $x = -1$. Checking our answer for $x = 2$ gives

$$\frac{x}{x+2} + \frac{2}{x-2} = \frac{x+6}{x^2-4}$$
$$\frac{2}{2+2} + \frac{2}{2-2} \stackrel{?}{=} \frac{2+6}{2^2-4}$$
$$\frac{2}{4} + \frac{2}{0} \stackrel{?}{=} \frac{8}{0}$$

This answer is undefined.

Checking our answer for $x = -1$ gives

$$\frac{x}{x+2} + \frac{2}{x-2} = \frac{x+6}{x^2-4}$$
$$\frac{-1}{-1+2} + \frac{2}{-1-2} \stackrel{?}{=} \frac{-1+6}{(-1)^2-4}$$
$$\frac{-1}{1} + \frac{2}{-3} \stackrel{?}{=} \frac{5}{-3}$$
$$-\frac{5}{3} = -\frac{5}{3}$$

This is correct. The solution to the equation is $x = -1$.