

# 15.4

## teacher notes

### Objectives

- Add rational expressions.
- Subtract rational expressions.

$$\Omega \frac{1}{15750}$$

$$\Delta = .00 \pi + \frac{1}{200000 \sqrt{xy}}$$

$$5-6 \mid \sqrt{xy} \frac{1}{12} \Delta$$

### Prerequisites

Adding and subtracting fractions  
Factoring polynomials  
Simplifying rational expressions  
Combining like terms  
Multiplying monomials and binomials

### Get Started

- Break the class into pairs. Give each pair two fractions and ask them to find the LCD. Some good choices include:  $\frac{5}{12}$  and  $\frac{7}{18}$ ,  $\frac{1}{7}$  and  $\frac{5}{6}$ ,  $\frac{1}{6}$  and  $\frac{19}{45}$ .
- Have each pair write their results on the board, describing the process they used to find the LCD. The LCD's in the examples given above are 36, 42, and 90, respectively.
- Ask each pair to list the prime factorization of the denominators in their original pair of fractions and in their LCD, in exponential form.  
Responses to the example here are:
 

$12 = 2^2 \cdot 3$	$18 = 2 \cdot 3^2$	$36 = 2^2 \cdot 3^2$
$7 = 7$	$6 = 2 \cdot 3$	$42 = 2 \cdot 3 \cdot 7$
$6 = 2 \cdot 3$	$45 = 3^2 \cdot 5$	$90 = 2 \cdot 3^2 \cdot 5$
- Lead a classroom discussion to observe the results of the prime factorization and conclude that the LCD contains each different factor listed in either denominator, and for factors listed in both denominators, the larger power of that factor is chosen.

### Vocabulary

Rational expression (Lesson 15-1)  
Least common denominator  
(Lesson 1-3)  
Factor (Lesson 1-3)  
Like terms (Lesson 2-4)

# Section 1

## Expand Their Horizons

In Section 1, students will add and subtract rational expressions with like, or common, denominators. Rational expressions are fractions. To add or subtract rational expressions with common denominators, add or subtract their numerators and keep the denominator the same. Simplify, if necessary. The work formula is introduced through a variation of the distance formula  $d = rt$ . The amount of work done is equal to the rate at which work can be done multiplied by the time it takes to complete the work. In an alternate form, time equals work divided by rate.

Consider the expression  $\frac{5}{6x} + \frac{7}{6x}$ . As both rational terms have the same denominator, the numerators will be added as the denominators remain the same:  $\frac{5+7}{6x}$  or  $\frac{12}{6x}$ . Both the numerator and denominator of the result are monomials; the rational expression can be immediately reduced by the common factor six to yield  $\frac{2}{x}$ .

Consider the expression  $\frac{2A-7}{6A} + \frac{A+4}{6A}$ . Common denominators indicate this expression can be added by simply adding the numerators and keeping the denominators the same. Because the numerators are polynomials, parentheses are used:  $\frac{(2A-7) + (A+4)}{6A}$ . To add polynomials, parentheses may be dropped and like terms added; this expression becomes  $\frac{3A-3}{6A}$ . To simplify, the numerator may be factored. Factorization yields  $\frac{3(A-1)}{6A}$ . The fraction can be reduced by dividing out the common factor three to get the final result  $\frac{A-1}{2A}$ .

Consider the expression  $\frac{5y+8}{2y+6} - \frac{y-4}{2y+6}$ . The denominators are the same; thus, the numerators must be subtracted while the denominators remain as they are. Because the numerators are polynomials to be subtracted, when placing them together in

one fraction, each term should be placed in parenthesis:  $\frac{(5y+8) - (y-4)}{2y+6}$ . To subtract polynomials and remove the parentheses, the first polynomial stays the same; the subtraction operation is changed to addition, and the second polynomial is multiplied by negative one. This yields  $\frac{5y+8 + (-1)(y-4)}{2y+6}$ .

Applying the Distributive Property, the rational expression becomes  $\frac{5y+8-y+4}{2y+6}$ , combining like terms, this gives  $\frac{4y+12}{2y+6}$ . Once the difference is found, the fraction may be simplified. To simplify, recall the binomials in the numerator and the denominator must first be factored. This factorization, removing the common factor four in the numerator and the common factor two in the denominator, is  $\frac{4(y+3)}{2(y+3)}$ . Reduce the common factors of two and  $(y+3)$ ; the result is two.



### Common Error Alert

In subtraction problems with more than one term in the subtrahend (term being subtracted or second term), make certain students subtract the entire polynomial by distributing negative one through the subtrahend as in the previous example. Students, who fail to do this, often subtract the first term but add the remaining terms due to this sign error. Review, if necessary.



The denominators are the same; add the numerators and put the sum over the common denominator:  $\frac{x+1}{x+1}$ . Since the numerator has no like terms, nothing can be combined. Both the numerator and denominator have the common factor  $(x+1)$ , which can be divided out to get  $\frac{1}{1}$  or just one.

## Additional Examples

**1. Simplify:**  $\frac{2(3x - 9)}{x} - \frac{3(2x - 6)}{x}$ .

The denominators are the same; thus, the numerators must be subtracted, and the denominator will remain the same. Placing them as one fraction gives  $\frac{2(3x - 9) - 3(2x - 6)}{x}$ . The Distributive Property must be applied; multiply the second binomial by negative three. This yields  $\frac{6x - 18 - 6x + 18}{x}$ . Combining like terms,  $6x$  and  $-6x$  become zero and  $-18$  and  $18$  become zero. The expression  $\frac{0}{x}$  equals zero.

**2. Simplify:**  $\frac{x^2 + 4x}{x^2 + 8x + 15} + \frac{3x + 10}{x^2 + 8x + 15}$ .

The denominators are the same; thus, the numerators may be added while the denominator remains the same. Adding the numerators gives  $\frac{(x^2 + 4x) + (3x + 10)}{x^2 + 8x + 15}$ . Simplification requires adding like terms. The sum of these fractions becomes  $\frac{x^2 + 7x + 10}{x^2 + 8x + 15}$ . Once the sum has been found, the trinomials must be factored to determine if the rational expression can be reduced. Factoring reveals  $\frac{(x + 2)(x + 5)}{(x + 3)(x + 5)}$ . A common factor of  $x + 5$  is found in the numerator and denominator. Dividing out this common factor, the resulting sum in reduced form is  $\frac{x + 2}{x + 3}$ .

## Section 2

### Expand Their Horizons

In Section 2, students will add and subtract rational expressions with unlike denominators. In this case, the least common denominator (LCD) must be found. Then, all fractions must be written as equivalent fractions with this LCD. Finally, the rational terms are added or subtracted as those with common denominators; the numerators are added or subtracted, and the denominator, which is now the LCD, remains the same.

To find the LCD, list all factors of each denominator raised to the appropriate powers. The LCD is the product of all factors of each denominator; for factors common to each denominator, the factor containing the highest power is used. Consider the LCD of  $\frac{5}{24} + \frac{1}{63}$ . The denominators, written as the products of primes, are  $24 = 2^3 \cdot 3$  and  $63 = 3^2 \cdot 7$ . The LCD must contain two, three, and seven because these are the prime factors within the denominators. The factor two is only contained within the denominator 24, where it has a power of three; therefore,  $2^3$  is part of the LCD. The factor three is contained in 24,

with a power of one, and in 63, with a power of two; because two is the higher power,  $3^2$  is a part of the LCD. The factor seven is only contained within the denominator 63, where it has a power of one; thus  $7^1$  or just seven is the last part of the LCD. The LCD is  $2^3 \cdot 3^2 \cdot 7$ , which is 504.

Consider  $\frac{2}{3x} + \frac{1}{x}$ . The denominator  $3x$  has two factors, three and  $x$ . The denominator of the second fraction has only the single factor,  $x$ . The LCD will be  $3x$ . All factors are contained in  $3x$ , and thus,  $3x$  is a multiple of each. The first fraction already has this denominator; thus, no change needs to be made. The second fraction, however, must be written as an equivalent fraction with the denominator  $3x$ . To make an equivalent fraction, the numerator and the denominator must be multiplied by the same number (which has the effect of multiplying by one and thus, not changing the value of the fraction). For the denominator  $x$  to equal the LCD  $3x$ , it must be multiplied by three. Therefore, the numerator must also be multiplied by three, yielding  $\frac{1 \cdot 3}{x \cdot 3}$  or  $\frac{3}{3x}$ . The expression is now  $\frac{2}{3x} + \frac{3}{3x}$ , which can be added as  $\frac{2+3}{3x}$  and finally  $\frac{5}{3x}$ .



### Common Error Alert

Students may try to reduce fractions after finding the LCD but before the sum or difference has been found. This will reduce the fractions to their original form and eliminate the LCD.

Consider the addition  $\frac{x}{5} + \frac{3}{2x}$ . The LCD must be  $10x$ . If the denominators have no common factors, the LCD is the product of those denominators; here,  $5 \cdot 2 \cdot x = 10x$ , and  $10x$  will be the LCD. To find the equivalent fractions with this LCD, determine what factor must be multiplied to each denominator to create the LCD. For the denominator five, a factor of  $2x$  is needed to get the product  $10x$ ; thus, the numerator and denominator of that fraction must be multiplied by  $2x$ :  $\frac{x \cdot 2x}{5 \cdot 2x} = \frac{2x^2}{10x}$ . For the denominator  $2x$ , a factor of five is needed to make the product  $10x$ ; thus, the numerator and denominator of that fraction must be multiplied by five:  $\frac{3 \cdot 5}{2x \cdot 5} = \frac{15}{10x}$ . Adding these rational terms gives  $\frac{2x^2 + 15}{10x}$ .

Consider the difference  $\frac{3}{x^2 - 25} - \frac{2}{x^2 + 9x + 20}$ . To determine the LCD, the denominators must be factored:  $\frac{3}{(x+5)(x-5)} - \frac{2}{(x+4)(x+5)}$ . The LCD must contain the highest power of each different factor; thus, the LCD is  $(x+5)(x-5)(x+4)$ . Keep the LCD in factored form; factored form is needed to determine the equivalent fractions. To obtain the LCD, the first fraction must be multiplied by the factor  $(x+4)$  in its numerator and denominator, and the second fraction must be multiplied by the factor  $(x-5)$  in its numerator and denominator. The expression becomes  $\frac{3(x+4)}{(x+5)(x-5)(x+4)} - \frac{2(x-5)}{(x+4)(x+5)(x-5)}$ . Using the Distributive Property with the numerators, this becomes  $\frac{3x+12}{(x+5)(x-5)(x+4)} - \frac{2x-10}{(x+4)(x+5)(x-5)}$ . Writing this as one fraction, the expression is  $\frac{(3x+12) - (2x-10)}{(x+5)(x-5)(x+4)}$ . Recall that to subtract polynomials and remove the parentheses, the first polynomial remains the same; subtraction is changed to addition, and negative one is multiplied throughout the second polynomial. The result is  $\frac{3x+12+(-2x)+10}{(x+5)(x-5)(x+4)}$ . The last step

is to combine like terms, which yields

$\frac{x+22}{(x+5)(x-5)(x+4)}$  or  $\frac{x+22}{x^3+4x^2-25x-100}$  if the denominators are multiplied together.

In the last example, once equivalent fractions are obtained, the expression can be written as one fraction,  $\frac{3(x+4) - 2(x-5)}{(x+5)(x-5)(x+4)}$ . Then the Distributive Property can be applied, distributing negative two throughout the second polynomial to get  $\frac{3x+12-2x+10}{(x+5)(x-5)(x+4)}$ . This achieves the same result as before but requires one less step, because distributive multiplication by negative one to change the subtraction to addition is incorporated within another step.



Neither denominator can be factored—both are prime. Thus, the LCD, which must contain each factor of the denominators, is the product of these denominators:  $(a+2)(a-2)$ . Changing each to an equivalent fraction with this LCD, the expression becomes  $\frac{4(a-2)}{(a+2)(a-2)} - \frac{3(a+2)}{(a-2)(a+2)}$ . Changing the subtraction to addition, and changing three to negative three, the Distributive Property makes the expression  $\frac{4a-8-3a-6}{(a+2)(a-2)}$ . Combining like terms yields  $\frac{a-14}{(a+2)(a-2)}$  or  $\frac{a-14}{a^2-4}$ , which is the final result since the numerator cannot be factored, and there are no common factors to divide out.

### Look Beyond

Finding common denominators for the purpose of combining terms only begins here. In trigonometry, the denominator of a rational term might be an entire function (like sine, cosine, or tangent) to combine with another such term. Yet, the technique is the same: factor the denominators, create an LCD which includes all the factors, and make sure to multiply the numerator by the same factor used to multiply the denominator. Common denominators are found from arithmetic, to algebra, to trigonometry, and beyond!

## Additional Examples

**1. Simplify:**  $\frac{3}{x-5} + \frac{1}{5-x}$ .

These fractions do not have a common denominator. However, multiplying negative one by the denominator of the second fraction would change it from  $5 - x$  to  $x - 5$ . Any factor multiplied to the denominator must also be multiplied to the numerator to produce an equivalent fraction. Thus the expression becomes  $\frac{3}{x-5} + \frac{1 \cdot -1}{(5-x) \cdot -1}$ . Simplifying, this becomes  $\frac{3}{x-5} + \frac{-1}{x-5}$  and finally,  $\frac{2}{x-5}$ . Notice  $5 - x$  could have been used as the LCD; in which case, the first fraction would have been multiplied by  $\frac{-1}{-1}$ , and the result would have been  $\frac{-2}{5-x}$ . This is an equivalent fraction to the other answer.

**2. Simplify:**  $\frac{2}{x^2+4x+4} - \frac{2}{x^2-4}$ .

To determine the LCD, both denominators must be factored:  $\frac{2}{(x+2)^2} - \frac{2}{(x+2)(x-2)}$ . The LCD must contain each factor to the higher power given; the LCD is  $(x+2)^2(x-2)$ . The numerator of the first fraction must be multiplied by  $x-2$  to get an equivalent fraction with this LCD; the numerator of the second fraction must be multiplied by  $x+2$  to do the same. The expression becomes  $\frac{2(x-2) - 2(x+2)}{(x+2)^2(x-2)}$ . Distributing, and paying careful attention to the negative two on the second polynomial of the numerator, this becomes  $\frac{2x-4-2x-4}{(x+2)^2(x-2)}$ . Combining the like terms  $2x$  and  $-2x$  yields zero, but  $-4$  and  $-4$  yield  $-8$ . The result is  $\frac{-8}{(x+2)^2(x-2)}$ .

