15.3 teacher notes

Objectives

- Multiply rational expressions.
- Divide rational expressions.
- Simplify complex fractions.

Vocabulary

Restricted value (Lesson 15-1)
Factor (Lesson 1-3)
Common factor (Lesson 12-1)

Prerequisites

Multiplying fractions

Dividing fractions

Factoring polynomials

Simplifying rational expressions

Multiplying monomials and binomials

Get Started

- Break class into two groups. Assign the first group several numbers to divide by two. For example, give the group the numbers six, 10, and 15 and have them divide each by two.
- Assign the second group the same numbers, this time to multiply by $\frac{1}{2}$.
- Have one student from each group show their work with solutions. The results from each group should be the same. For the numbers given here, solutions are three, five, and $7\frac{1}{2}$.
- Discuss why the results are the same. Lead students to conclude that division is the same as multiplication by the reciprocal. This concept will be used in today's lesson.

Section 1

Expand Their Horizons

In Section 1, students will apply the rules for multiplying fractions to larger rational expressions. The property for fraction multiplication states that $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ for $b, d \neq 0$. That is, fractions are multiplied together by multiplying the numerators and multiplying the denominators, provided neither denominator is equal to zero (which would make the fraction undefined). Throughout this lesson, denominators will not be equal to zero.

Recall when multiplying fractions, the fractions can be reduced before or after the product is found. Consider the example $\frac{3}{4} \cdot \frac{20}{33}$. The product can be found first, $\frac{60}{132}$, and then, the numerator and denominator reduced by the common factor 12 to get $\frac{5}{11}$. Alternatively, common factors of three and four may be divided out of the existing fractions, as $\frac{{}^{1}\cancel{3}}{{}_{1}\cancel{4}} \cdot \frac{\cancel{20}^{5}}{\cancel{33}_{11}}$ to get the same fraction $\frac{5}{11}$. The latter is employed for multiplying rational expressions involving variables.

Consider the expression $35x^3 \cdot \frac{y}{9y^4} \cdot \frac{12}{10x^2}$. To multiply, make all terms into fractions; for $35x^3$, this is $\frac{35x^3}{1}$. Placing any quantity over one changes it to an equivalent fraction. Then, divide out common factors. The common factor five can be divided into 35 and 10 leaving seven and two respectively. The common factor three can be divided into nine and 12 leaving three and four respectively. Recall that the common factor for like variable terms is the common variable with the lower exponent. The common factor x^2 can be divided out of x^3 and x^2, leaving x and one respectively. The common factor y can be divided out of y and y^4, leaving one and y^3. Finally, the four and two, which remain from previously reduced factors, can further be reduced by the common factor of two to get two and one respectively. Notice the common factors were found within the same fraction, between adjacent fractions, or between non-adjacent fractions; the common factor merely needs to be found in a numerator and a denominator. It should be pointed out the order in which common factors are reduced is not important and will give the same result. Finally, what remains is $\frac{{}^{7}\cancel{35}x^{\cancel{3}\,1}}{1} \cdot \frac{\cancel{y}^{1}}{{}_{3}\cancel{9}\cancel{y^4}_{y^3}} \cdot \frac{\cancel{12}^{\cancel{4}\,2}}{{}_{1}\cancel{2}\,\cancel{10}\cancel{x^2}}$. Multiplying the remaining factors gives $\frac{7 \cdot x \cdot 1 \cdot 2}{1 \cdot 3 \cdot y^3 \cdot 1 \cdot 1}$, which is $\frac{14x}{3y^3}$.

Common Error Alert

Students may attempt to convert non-variable factors of a rational expression to a mixed number; in the previous problem, $\frac{14}{3}$ equals $4\frac{2}{3}$. This gives $4\frac{2}{3}\left(\frac{x}{y^3}\right)$. Although this is mathematically correct, it is hard to read. It is not immediately apparent $\frac{2}{3}$ is not a factor in this expression. Rational expressions should be reduced but not converted to another form.

Consider the example $\frac{5}{3} \cdot \frac{3x}{10x}$. The common factor of five can be divided out of five and 10 to leave one and two. The common factor of three and three is three, leaving one and one when reduced. Finally, the factors x and x leave one and one when the common factor x is divided out. After reducing, the multiplication becomes $\frac{{}^{1}\cancel{5}}{{}_{1}\cancel{3}} \cdot \frac{{}^{1}\cancel{3}\cancel{x}^{1}}{{}_{2}\cancel{10}\cancel{x}_{1}}$ which becomes $\frac{1}{2}$.

Common Error Alert

When the only remaining factor(s) is(are) in the numerator, the denominator may be removed because division by one is indicated and division by one yields the original expression. However, when the only remaining factor(s) is(are) in the denominator, a one must remain in the numerator—this cannot be reduced. Consider the examples below, noting $\frac{2}{1} = 2$, but $\frac{1}{2} \neq 2$:

$$\frac{5}{3} \cdot \frac{3x}{10x} = \frac{{}^{1}\cancel{5}}{{}_{1}\cancel{3}} \cdot \frac{{}^{1}\cancel{3}\cancel{x}^{1}}{{}_{2}\cancel{10}\cancel{x}_{1}} = \frac{1 \cdot 1 \cdot 1}{1 \cdot 2 \cdot 1} = \frac{1}{2}$$

$$\frac{3}{5} \cdot \frac{10x}{3x} = \frac{{}^{1}\cancel{3}}{{}_{1}\cancel{5}} \cdot \frac{{}^{2}\cancel{10}\cancel{x}^{1}}{{}_{1}\cancel{3}\cancel{x}_{1}} = \frac{1 \cdot 2 \cdot 1}{1 \cdot 1 \cdot 1} = 2.$$

While reducing fractions is sometimes referred to as canceling, it may lead to the mistake of omitting a numerator of one when it is needed.

For rational expressions with monomial numerators and denominators, common factors may be immediately divided out. However, for rational expression containing binomials or trinomials, the factors must be determined before they can be divided out. The only difference between these and the previous rational products is each polynomial must first be factored before the common factors can be reduced. The steps for multiplying rational expressions with binomials and trinomials are:

(1) Factor each polynomial,
(2) Cancel common factors,
(3) Multiply remaining factors in the numerator, and
(4) Multiply remaining factors in the denominator.

Common Error Alert

After factorization and simplification of an expression, if there remains an expression like $\frac{1}{(x+5)(x+5)}$, then these factors do not cancel. The expression $\frac{1}{(x+5)(x+5)}$ equals $\frac{1}{(x+5)^2}$.

Consider the expression $\frac{2x^5}{x+2} \cdot \frac{x^2-3x-10}{6x^3-30x^2}$. The numerator of the first fraction is a monomial; factoring is not needed. The denominator of the first fraction is a binomial, but it cannot be factored. However, the fraction bar is a grouping symbol as well as a division symbol, so this denominator will be written using parenthesis as $(x + 2)$. This notation was not necessary in previous problems because all were monomials that could be separated into factors. The numerator of the second fraction is a trinomial that factors as $(x + 2)(x - 5)$. Finally, the denominator of the second fraction can be factored by taking out the common factor, $6x^2(x - 5)$. The fraction multiplication becomes $\frac{2x^5}{(x+2)} \cdot \frac{(x+2)(x-5)}{6x^2(x-5)}$. The two and six have a common factor of two that may be divided out; the x^5 and x^2 have a common factor of x^2 that may be divided out. The factors $(x + 2)$ and $(x - 5)$ have common factors that may be divided out. The remaining factors in the numerator are one, x^3, one, and one; multiplied together this gives a new numerator of x^3. The remaining factors in the denominator are one, three, one, and one; multiplied together this gives a new denominator of three. The result is $\frac{x^3}{3}$.

1 To begin, place the numerator of the first fraction in parenthesis to show the grouping represented by the fraction bar; this term cannot be factored. Next, write $(3t + 18)$ as a fraction by placing it over one and then, remove the common factor three to get $3(t + 6)$. Factor the denominator of the first fraction by first removing the common factor nine to get $9(t^2 + 7t + 6)$ and then, by factoring the trinomial as $9(t + 1)(t + 6)$. The expression becomes $\frac{(t+6)}{9(t+1)(t+6)} \cdot \frac{3(t+6)}{1}$, which reduces to $\frac{t+6}{3(t+1)}$ by dividing out the common factor three into nine and three and by dividing the common factor $(t + 6)$ into that term in the denominator with either in the numerator but not both. The final answer can also be written as $\frac{t+6}{3t+3}$, with the factors of the denominator being multiplied together.

Common Error Alert

Remind students values within grouping symbols may not be canceled between the numerator and the denominator. The expression $\frac{(x+2)}{(x+6)}$ is in simplest form. This expression does not equal $\frac{x}{x+3}$ by canceling a common factor of two between two and six.

Additional Examples

1. Simplify: $\frac{6}{5y} \cdot \frac{x^2y^3}{3z^4} \cdot \frac{10z^6}{4x^5y^2}$.

Because all numerators and denominators are monomials, factoring is not necessary.

$$\frac{6}{5y} \cdot \frac{x^2y^3}{3z^4} \cdot \frac{10z^6}{4x^5y^2} = \frac{1}{y} \cdot \frac{x^2y^3}{z^4} \cdot \frac{z^6}{x^3y^2}$$
$$= \frac{1}{1} \cdot \frac{1}{z^4} \cdot \frac{z^6}{x^3}$$
$$= \frac{1}{1} \cdot \frac{1}{1} \cdot \frac{z^2}{x^3}$$
$$= \frac{z^2}{x^3}$$

2. Simplify: $\frac{x^2 + 6x + 8}{x + 3} \cdot \frac{5x + 15}{10x + 40}$.

Factor the numerator of the first factor as $(x + 2)(x + 4)$.

$$\frac{x^2 + 6x + 8}{x + 3} \cdot \frac{5x + 15}{10x + 40} = \frac{(x + 2)(x + 4)}{x + 3} \cdot \frac{5(x + 3)}{10(x + 4)}$$
$$= \frac{x + 2}{1} \cdot \frac{1}{2}$$
$$= \frac{x + 2}{2}$$

Section 2

Expand Their Horizons

In Section 2, students will divide rational expressions. Dividing with fractions is defined as multiplying by the reciprocal of the divisor: $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$ for b, c, $d \neq 0$. In the same way, division with rational expressions is defined as multiplication of the first expression by the reciprocal of the second expression. Some may call the reciprocal an inverted fraction. Once written as a multiplication problem, rules for multiplying rational expressions apply as in Section 1.

Consider the expression $\frac{x^2 - 4}{x^2 + 7x + 10} \div \frac{x^2 - 8x + 12}{x^2 - x - 30}$. First, the division is written as multiplication by the reciprocal of the divisor: $\frac{x^2 - 4}{x^2 + 7x + 10} \cdot \frac{x^2 - x - 30}{x^2 - 8x + 12}$. As with all multiplication problems involving binomials and trinomials, all parts of both fractions must be factored: $\frac{(x + 2)(x - 2)}{(x + 2)(x + 5)} \cdot \frac{(x + 5)(x - 6)}{(x - 2)(x - 6)}$. All factors divide out, leaving one as the result.

Recall that a fraction whose numerator or denominator includes another fraction is a complex fraction. A complex fraction is merely the representation of a division problem and thus, can be simplified as such. Consider the complex fraction $\frac{\frac{2x + 6}{2}}{x^2 - 9}$. Written as a division, this becomes $\frac{2x + 6}{2} \div (x^2 - 9)$. Because $(x^2 - 9)$ can be placed over one to become a fraction, when this expression is rewritten as multiplication by the reciprocal, it becomes $\frac{2x + 6}{2} \cdot \frac{1}{x^2 - 9}$. Factoring, the expression is $\frac{2(x + 3)}{2} \cdot \frac{1}{(x + 3)(x - 3)}$ where the 2's and the $(x + 3)$'s are divided out, leaving $\frac{1}{x - 3}$ as the simplification of this complex fraction.

2 Leave the first fraction as it is, change the operation to multiplication, and invert the second fraction. This yields $\frac{4}{5s^2} \cdot \frac{6s^2 + 9s}{7}$. Factoring the numerator of the second fraction, the expression becomes $\frac{4}{5s^2} \cdot \frac{3s(2s + 3)}{7}$; the other parts of the fractions do not need to be factored; they are monomials. The only common factor to be divided out is s, leaving $\frac{4}{5\cancel{s}_s\cancel{^2}} \cdot \frac{3\cancel{s}^1(2s + 3)}{7}$, which becomes $\frac{12(2s + 3)}{35s}$ or $\frac{24s + 36}{35s}$.

Look Beyond

The concept behind simplifying multiplication and division of rational expressions involves breaking down the problem into manageable bits, a concept utilized throughout mathematics. Specifically, the concept of dividing out common factors will be used for solving rational expressions in the next module.

Additional Examples

1. Simplify: $\frac{x^2 - 16}{x + 3} \div \frac{x + 4}{x^2 - 9}$.

First, rewrite the division as multiplication with the reciprocal of the divisor or second fraction: $\frac{x^2 - 16}{x + 3} \cdot \frac{x^2 - 9}{x + 4}$. The binomials $x + 3$ and $x + 4$ cannot be factored and can be placed in parenthesis. The other binomials are sum and difference binomials, which are factored as $(x + 4)(x - 4)$ and $(x + 3)(x - 3)$. The expression becomes $\frac{(x + 4)(x - 4)}{(x + 3)} \cdot \frac{(x + 3)(x - 3)}{(x + 4)}$, with the $(x + 4)$ and $(x + 3)$ factors dividing out to leave 1's. The final result is $(x - 4)(x - 3)$, which multiplies through FOIL to become $x^2 - 7x + 12$. Since the denominator is one, it may be omitted.

2. Simplify: $\dfrac{\frac{x^2 + 6x + 8}{x + 4}}{x + 2}$.

A complex fraction can be rewritten as the division of the top fraction by the bottom. Because division is then rewritten as multiplication with the reciprocal of the divisor, a complex fraction can be rewritten directly as a multiplication problem in which the top fraction remains the same, and the bottom fraction is inverted. Since the denominator is now $\frac{(x + 2)}{1}$, the expression becomes $\frac{x^2 + 6x + 8}{x + 4} \cdot \frac{1}{(x + 2)}$. The denominators are prime, and only the numerator of the first fraction factors with the expression becoming $\frac{(x + 2)(x + 4)}{x + 4} \cdot \frac{1}{(x + 2)}$. All factors divide out becoming 1's, so the simplification of this complex fraction is one.