15.1 teacher notes

Objective

- Find the restrictions that must be placed on the variable in a rational expression.

Vocabulary

Rational expression
Domain of an expression
Restricted value
Factor (Lesson 12-1)
GCF (Lesson 12-1)

Prerequisites

Solving linear, quadratic, and cubic equations

Get Started

- Say, "Imagine a class is having a pizza party. They have ordered a certain number of slices, s, and they have a certain number of people, p, to feed. Write an expression to show the number of slices each person can get." The correct answer is $s \div p$. On the board, write the answer as a ratio: "number of slices each $= \frac{s}{p}$."
- Say, "There are 20 slices and four people. What expression shows the number of slices each person can get?" $\frac{s}{p} = \frac{20}{4} = 5$ slices per person. Repeat the question for other convenient values of s and p (e.g. 12 slices and six people; 30 slices and 15 people).
- Next, say, "What if there are zero slices of pizza and 10 people? What expression shows the number of slices each person gets?" $\frac{s}{p} = \frac{0}{10} = 0$ slices per person.
- Finally, "What if there are 15 slices of pizza and zero people? What expression shows the number of slices each person gets?" $\frac{s}{p} = \frac{15}{0}$. The class should carefully considers the question. Guide them to see there is no meaningful answer—the question does not make sense.
- Say, "In this lesson, we will study algebraic fractions, and we will make sure the variables never take on a value that causes the denominator to be zero."

Section 1

Expand Their Horizons

In this lesson, rational expressions are introduced. A rational expression is essentially a division problem expressed as a fraction. In a rational expression, both the numerator and the denominator are polynomials.

Review the definition of polynomial with students. A polynomial is the sum or difference of monomials. Monomials are the products of constants, of variables, or of constants and variables. So, the expression $\frac{4}{x-3}$ is a rational expression since it shows the monomial four in the numerator and the binomial $x - 3$ in the denominator.

Common Error Alert

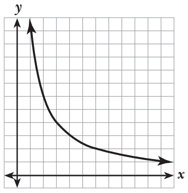
Some students may not consider the constant four a polynomial. Certain polynomials have special names. A monomial is a polynomial of just one term. A binomial is a polynomial consisting of two monomials. A trinomial is a polynomial comprised of three monomials. A constant is a monomial with no variables; it is considered a special polynomial known as a *constant polynomial*.

It is essential students understand the idea that division by zero is undefined. It is worthwhile to spend some time emphasizing this point.

Students sometimes confuse dividing by zero with dividing zero by a number. It is legitimate to divide zero by a number. That is, the numerator of a fraction can be zero. This is exemplified in the Get Started activity when zero pizzas are divided among 10 people. Each person gets no pizza. Zero divided by any nonzero number is always zero.

It is division by zero that is not allowed. There are several ways to help students see why division by zero is undefined. The Get Started activity provides one common-sense explanation. Another way is to change the division problem to a multiplication problem. For example, $\frac{12}{3} = 4$ can be written as $12 = 3 \cdot 4$. When the denominator is zero, $\frac{5}{0} = n$ becomes $5 = 0 \cdot n$. For what values of n is this equation true? There are no values because any number times zero is zero. That is why division by zero does not make sense and is said to be undefined.

Visual learners may benefit from a graphical explanation. Ask them to determine and plot some ordered pairs for the function $y = \frac{12}{x}$. Plot the points (12, 1), (6, 2), (4, 3), (3, 4), (2, 6), and (1, 12). Point out, as the number in the denominator (x) gets smaller, the value of the expression $\frac{12}{x}$ gets bigger. Continue plotting points with fractional values of x. When $x = \frac{1}{2}$, $y = 24$. When $x = \frac{1}{4}$, $y = 48$. When $x = \frac{1}{100}$, $y = 1{,}200$. As smaller and smaller values of x are chosen, larger and larger outputs are produced until the output becomes infinitely large. That is, as input values get infinitesimally close to zero, output values become infinitely large.



On the graph, the graph gets infinitely “higher” as x values get smaller. Some people say the function “explodes” at zero. At zero, the output is indeterminately large; it is undefined.

For this reason, when variables appear in the denominator of a rational expression, the variable must be restricted so that it never takes on a value which causes the denominator to be zero.

To find the restricted values of the variable, set the denominator equal to zero and then, solve the resulting equation. For instance, in the expression $\frac{1}{x-5}$, set the denominator equal to zero. If $x - 5 = 0$, then $x = 5$. The value(s) that make the equation true, in this case five, become(s) the restrictions on the variable. The variable cannot take on these value(s).

In this lesson, restrictions on variables are described in two different ways. One is simply to list the restricted values. In the above example, the restricted value is five. The other way is to describe the variable's domain—the set of values the variable can take on. Express the domain by writing inequalities to show the restricted values. An inequality describing the restricted value for the example above would be $x \neq 5$. Offer the idea that the symbol "$\neq$" could be read as "cannot equal" (instead of "does not equal") for this lesson.

1 The variable x is the only term in the denominator; it can never have a value of zero. So, x cannot equal zero. The restricted value is zero.

2 In the previous question, it was determined x cannot equal zero. The domain of the variable is the set of all real numbers less the case where x is zero.

Common Error Alert

Students may incorrectly write the domain as $x = 0$. Remind them the domain is all real numbers except those listed.

3 To determine restrictions on x, set the denominator equal to zero and solve for x. If $x + 9 = 0$, then $x = -9$. The restricted value is -9. The domain of the variable is $x \neq -9$.

Common Error Alert

For the expression $\frac{4}{x+9}$, students may say the restriction is $x \neq 0$. Show them that if $x = 0$, then the denominator equals nine which is a legitimate divisor. The value of x *can* equal zero provided the value of the entire expression is in the denominator, $x + 9$, which does not equal zero. Demonstrate that when $x = -9$, $x + 9 = 0$, and the expression is undefined so negative nine is a restricted value.

4 Set the denominator equal to zero; then, solve for n:

$$\begin{aligned} 2n - 6 &= 0 \\ 2n &= 6 \\ \frac{2n}{2} &= \frac{6}{2} \\ n &= 3 \end{aligned}$$

The restricted value is three.

Additional Examples

Find any restricted values of the variable.

1. $\frac{x - 4}{8}$

The value of the denominator in this expression is not dependent on the value of the variable; it is always eight. Since the numerator of an expression can take on any value, there are no restrictions on x. There are no restricted values of the variable.

2. $\frac{-2}{5 - 10x}$

Set the denominator equal to zero and then, solve.

$$\begin{aligned} 5 - 10x &= 0 \\ -10x &= -5 \\ x &= \frac{1}{2} \end{aligned}$$

The restricted value is $\frac{1}{2}$.

Section 2

Expand Their Horizons

In Section 2, the rational expressions have denominators that are polynomials with degree higher than one. The denominators in this section are quadratic and cubic polynomials.

Just as in Section 1, to find the restricted values of the variable, solve the equations that are created when the denominator is set equal to zero. In each problem in this lesson, the equation can be solved by factoring.

Students have learned to factor expressions by removing the GCF, and they learned to factor expressions of the form $ax^2 + bx + c$. Remind students that factoring the denominator of a rational expression uses this same process.

Point out to students, that in quadratic equations, there are either one or two restricted values. An example of a quadratic equation with one restricted value is $\frac{7}{x^2 + 4x + 4}$. The factors of denominator, $x^2 + 4x + 4$, are $(x + 2)$ and $(x + 2)$. The factors are identical; there is only one restricted value of –2. Similarly, cubic equations may have as many as three possible restricted values.

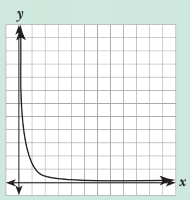
5 To solve the equation $x^2 + x - 6 = 0$, factor the trinomial. The factors are $(x - 2)$ and $(x + 3)$. When x is 2, $(x - 2)$ is zero. When x is –3, $(x + 3)$ is zero. So, x cannot equal 2, and x cannot equal –3. These are the restricted values for the domain of x.

Connections

The idea of restricted domains is not new. The domains of time and length are nonnegative. Negative values for these domains have no real-world application. The domain of the number of people populating planet Earth is the set of positive integers; the world cannot have partial people or negative people. Are there other situations where the domain (the set of possible values) is restricted?

Look Beyond

This lesson prepares students to study rational functions like $f(x) = \frac{1}{x}$. The graphs of such functions can approach, but never intersect, lines called *asymptotes*. An asymptote is a line that the graph of $f(x)$ approaches as x approaches a certain value or increases without bound.



For the graph of $f(x) = \frac{1}{x}$ shown above, the graph approaches the x-axis (but never touches it) as x gets infinitely large. Logically, the larger x gets, the smaller $\frac{1}{x}$ gets; yet, $\frac{1}{x}$ never becomes zero. For the function $f(x) = \frac{1}{x}$, the x-axis is an asymptote.

Also, for increasingly smaller positive values of x, $f(x)$ increases without bound, approaching the y-axis (but never touching it). Logically, the smaller x gets, the larger $\frac{1}{x}$ gets; yet, x can never equal zero. For the function $f(x) = \frac{1}{x}$, the y-axis is also an asymptote.

In future courses, students will learn how restrictions on the value of x can be of use in determining the equations of the asymptotes of a function's graph.

Common Error Alert

In Additional Examples Question 1, after factoring the denominator, the expression is $\frac{x - 3}{(x - 3)(x + 5)}$. The expression $(x - 3)$ appears in both the numerator and the denominator. Although $\frac{x - 3}{x^2 + 2x - 15}$ could be simplified to $\frac{1}{x + 5}$, because the original equation had two factors in the denominator, the restricted values remain 3 and –5.

Additional Examples

Find any restricted values of the variable.

1. $\dfrac{x - 3}{x^2 + 2x - 15}$

Set the denominator equal to zero; then, solve by factoring.

$x^2 + 2x - 15 = 0$

$(x - 3)(x + 5) = 0$

$x - 3 = 0$ or $x + 5 = 0$

$x = 3$ or $x = -5$

Domain: $x \neq 3$ and $x \neq -5$

2. $\dfrac{4}{x^3 - 36x}$

Set the denominator equal to zero; then, solve by factoring. Two factoring methods are necessary. Remove the common factor; then, factor the difference of squares.

$x^3 - 36x = 0$

$x(x^2 - 36) = 0$

$x(x - 6)(x + 6) = 0$

$x = 0$ or $x - 6 = 0$ or $x + 6 = 0$

$x = 0$ or $x = 6$ or $x = -6$

Domain: $x \neq 0$, $x \neq 6$, $x \neq -6$