13.6 teacher notes

Objective

- Solve problems using quadratic equations of one variable.

Vocabulary

Quadratic equation (Lesson 13-1)
Standard form of a quadratic equation (Lesson 13-1)
Quadratic formula (Lesson 13-5)

Prerequisites

Writing algebraic expressions for word phrases

Finding the area of a rectangle

Solving quadratic equations of one variable

Get Started

- Instruct students to sketch and to label a rectangle whose width is three units less than its length. Students should use whole numbers for each dimension.
- Ask students to calculate the area by using the formula $A = l \cdot w$. Students should write the area of their rectangle on the back of the paper.
- Ask students to exchange papers with a partner. Students should show their partner the side of their paper that has the area only.
- Instruct students to determine the dimensions of their partner's rectangle. Remind students the width of the rectangle is three units less than the length.
- Observe students as they work. If students need help getting started, suggest a *guess-and-check* method. They may start with two numbers that are three units apart, such as three and six, and test to see if the product equals the given area.
- After they have decided on the two numbers, tell students to check their answers by looking on the back of their partner's paper.
- Select two or three students who were successful to share their methods with the class.

Section 1

Expand Their Horizons

In Section 1, students will apply their skills in solving quadratic equations to solving rectangular area problems. In this lesson, students will need to solve quadratic equations of one variable using the methods previously taught in this module. Therefore, a brief review of solving quadratic equations by evaluating square roots, by factoring, by completing the square, and by using the quadratic formula is suggested prior to the start of this lesson.

The first step in solving rectangular area problems is converting the word phrases and sentences to algebraic expressions and equations. Then, through a series of substitutions, the algebraic expressions and equations are transformed into a quadratic equation in one variable. Finally, the quadratic equation is written in standard form and solved. Answers that do not make sense in the context of a problem are discarded.

In the first lesson example, students are provided with two pieces of information. They are told the length is two yards less than twice the width, and that the area of the rectangular field is 60 square yards. This information is written algebraically as $l = 2w - 2$ and $A = 60$. Students should also write the formula for the area of a rectangle, $A = l \cdot w$.

Next, the students must perform the necessary substitutions. Because $A = 60$, $A = l \cdot w$ is rewritten as $60 = l \cdot w$. Because $l = 2w - 2$, $60 = l \cdot w$ is rewritten as $60 = (2w - 2) \cdot w$ or $60 = (2w - 2)w$.

Common Error Alert

Emphasize to students they must use parentheses when they substitute an expression for length or width.

To solve the equation $60 = (2w - 2)w$, students must rewrite it in standard form. Distributing on the right side puts the equation in the form $60 = 2w^2 - 2w$. Subtracting 60 from each side puts the equation in standard form, $0 = 2w^2 - 2w - 60$.

Once the equation is in standard form, the students may choose any method of solving the equation to find the answer. The equation is solved by factoring in the lesson, but students may prefer to use the quadratic formula. Students may have a program on their calculators that perform the operations of the quadratic equation for them. Decide beforehand if these will be allowed, or in what capacity. Some teachers allow students to solve the equation as usual, showing their work on paper but using the calculator program as a check. Other teachers ban the use of these calculator programs altogether.

The quadratic equation $0 = 2w^2 - 2w - 60$ yields two answers, six and negative five. The width of a rectangle cannot be negative, so the solution $w = -5$ is discarded. The only reasonable solution is $w = 6$. The width of the rectangle is six yards.

Common Error Alert

Some students mistakenly believe that every negative solution must be discarded. This is incorrect because it depends on the situation. Some real world scenarios in which negative solutions would be feasible include: temperatures, stock market changes, business profits (negative profits are losses), and checking account/credit card balances. Encourage students to evaluate every solution for its reasonableness in the given problem before discarding it.

The problem asks for both dimensions, so the length of the rectangle must also be found. This is done by substituting six for w in an equation that has both variables, w and l. In the lesson, six is substituted into the equation $l = 2w - 2$, and the equation becomes $l = 2(6) - 2$. Then, $l = 12 - 2$, and $l = 10$ yards. However, six can also be substituted into the equation $60 = l \cdot w$. Then, $60 = l \cdot 6$ and $10 = l$.

Encourage students to review the answer presented on screen. Emphasize that answers must satisfy all of the requirements in a problem to be correct. A 6-yard by 10-yard rectangle will have an area of 60 square yards because $6 \cdot 10 = 60$. The length is two yards less than twice the width because $2(6) - 2 = 12 - 2 = 10$. Since *both* conditions are met, the answer is correct.

For additional practice in setting up quadratic equations, choose one of the student's rectangles from the opening activity. Tell the students the area of the rectangle and remind them the width was three less than the length. Students should write a quadratic equation in standard form that can be solved to find the dimensions of the rectangle.

If students are struggling, go through the following leading questions and instructions.

(In this example, the dimensions are five by eight.)

What is the area of the rectangle?	$A = 40$
How can you write that the width is three units less than the length algebraically?	$w = l - 3$
What is the formula for the area of a rectangle?	$A = l \cdot w$
Substitute the known area of the rectangle into the formula.	$40 = l \cdot w$
Use substitution again to rewrite the equation using only one variable.	$40 = l \cdot (l - 3)$
Distribute.	$40 = l^2 - 3l$
Put the equation in standard form.	$0 = l^2 - 3l - 40$

1 **Find the dimensions of a floor, given that the area of the floor is 96 square feet and the width of the floor is 12 feet less than three times the length.** Begin by writing the three equations: $A = 96$, $w = 3l - 12$, and $A = l \cdot w$. Then, use substitutions to write a quadratic equation of one variable.

$$A = l \cdot w$$
$$96 = l \cdot w$$
$$96 = l \cdot (3l - 12)$$

Write the equation in standard form.

$$96 = 3l^2 - 12l$$
$$0 = 3l^2 - 12l - 96$$

Solve the equation by factoring.

$$0 = 3(l^2 - 4l - 32)$$
$$0 = 3(l - 8)(l + 4)$$
$$l = 8 \text{ or } l = -4$$

Because length cannot be negative, the length of the floor must be eight feet. Substitute eight into an equation with two variables to find the width.

$$w = 3l - 12$$
$$w = 3(8) - 12$$
$$w = 24 - 12$$
$$w = 12$$

The width is twelve feet. The dimensions of the floor are eight feet by 12 feet.

Students may choose to use the quadratic formula to solve the equation. The solution is worked as follows:

$$l = \frac{12 \pm \sqrt{144 - 4(3)(-96)}}{6}$$
$$l = \frac{12 \pm \sqrt{1{,}296}}{6}$$
$$l = \frac{12 \pm 36}{6}$$
$$l = 8 \text{ or } l = -4$$

Additional Examples

1. Clint's rectangular swimming pool has an area of 240 square feet. The length is two feet more than half of the width. What are the dimensions of Clint's pool?

Use the information to write equations.

$A = 240 \quad l = \frac{1}{2}w + 2 \quad A = l \cdot w$

Use substitution to write a quadratic equation of one variable.

$240 = w\left(\frac{1}{2}w + 2\right)$

Write the equation in standard form.

$240 = \frac{1}{2}w^2 + 2w$

$0 = \frac{1}{2}w^2 + 2w - 240$

Solve by factoring.

$0 = 12\,(w^2 + 4w - 480)$

$0 = 12\,(w - 20)(w + 24)$

$w = 20$ or $w = -24$

The width is 20 feet. Find the length.

$l = \frac{1}{2}w + 2$

$l = \frac{1}{2}(20) + 2$

$l = 12$

The length is 12 feet. The dimensions of Clint's pool are 12 feet by 20 feet.

2. The area of a rectangle is 116 square meters. The length is $1\frac{1}{2}$ meters longer than twice the width. Find the dimensions of the rectangle.

In this solution, the equation is solved by completing the square.

$A = 116 \quad l = 2w + \frac{3}{2} \quad A = l \cdot w$

Use substitution to write a quadratic equation of one variable.

$116 = \left(2w + \frac{3}{2}\right)w = 2w^2 + \frac{3}{2}w$

Divide each term by two and complete the square.

$58 = w^2 + \frac{3}{4}w$

$58 + \left(\frac{3}{8}\right)^2 = w^2 + \frac{3}{4}w + \left(\frac{3}{8}\right)^2$

$58 + \frac{9}{64} = w^2 + \frac{3}{4}w + \frac{9}{64}$

Factor the right side and solve by evaluating square roots.

$\frac{3,721}{64} = \left(w + \frac{3}{8}\right)^2$

$\sqrt{\frac{3,721}{64}} = w + \frac{3}{8}$ or $-\sqrt{\frac{3,721}{64}} = w + \frac{3}{8}$

$\frac{61}{8} = w + \frac{3}{8}$ or $-\frac{61}{8} = w + \frac{3}{8}$

$w = \frac{58}{8} = 7\frac{1}{4}$ or $w = -\frac{64}{8} = -8$

The width is $7\frac{1}{4}$ meters. Find the length.

$A = l \cdot w \quad 116 + l \cdot 7\frac{1}{4} \quad 16 = l$

The length is 16 meters. The dimensions of the rectangle are 16 meters by $7\frac{1}{4}$ meters.

Section 2

Expand Their Horizons

In Section 2, students will use quadratic equations to solve vertical motion problems. Students will encounter these same types of problems later in physics.

In the lesson, Ferd plans to launch himself over a football field. The equation, which represents his height as a function of time, is given as $h = -16t^2 + 40t + 10$. Students may not understand why they should use the variables h and t instead of x and y in quadratic equations that model vertical motion. Remind them h represents height in feet, and t represents time in seconds. Since h stands for height and t stands for time, using these variables names can help avoid confusion.

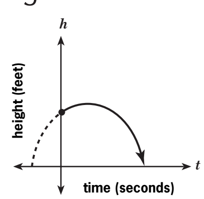
When height is represented in an equation as a function of time, the constant term indicates the initial height. The initial height is the height at which the object was launched. If the object was launched from ground level, the constant term is zero. If a person throws a ball, the initial height is the distance from the ground to the person's hand when the ball is released. In Ferd's equation, $h = -16t^2 + 40t + 10$, the constant term is 10, which means he will be launching himself from a point 10 feet above the ground.

The coefficient of the linear term indicates the velocity at takeoff. A ball thrown at a higher velocity will have a larger coefficient for t than one that is thrown at a lower velocity. In Ferd's equation, $h = -16t^2 + 40t + 10$, the coefficient of the linear term is 40, which means that Ferd plans on launching himself at an initial velocity of 40 feet per second.

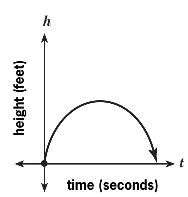
The coefficient of the squared term, –16, denotes the 1-G gravitational pull of the earth. This value will not change for the problems in this lesson.

In the lesson, Ferd wants to know how long he will be in the air. He will be in the air from the moment he takes off until the moment he hits the ground. When he hits the ground, his height is zero. So, zero is substituted for h, and the equation $h = -16t^2 + 40t + 10$ becomes $0 = -16t^2 + 40t + 10$. The quadratic formula is used to solve for t. As in Section 1, negative answers for time do not make sense, so those are discarded.

Draw a picture of the scenario to help students understand why the equation has two solutions, but only one is accepted. The graph is a parabola with two x-intercepts (t-intercepts), one positive and one negative, but only the part of the graph shown in Quadrant I represents Ferd's flight. Ferd is launching himself from the y-intercept (h-intercept). At this point, the time is zero, but the height is ten. Theoretically, there are two instances of when the graph has a height of zero, but following the path of Ferd's flight shows that he will land on the positive side. He cannot go backwards in time to the negative side of the graph. A graphing calculator shows this graph easily in the standard window. Remember to use the associated equation $y = -16x^2 + 40x + 10$ with the graphing calculator.



When the initial height is zero, however, the parabola looks like the following:



The two moments when the height is zero are at take-off and at landing.

In the lesson, it is calculated that Ferd would be in the air for about 2.73 seconds. It should be noted, however, that in reality the time might be a bit different because the formula assumes there is no air resistance.

2 **Find the time required for a ball to reach a height of 50 feet when the height of the ball is given by the equation $h = -16t^2 + 5t + 100$.** To find the elapsed time, substitute 50 for the height and solve for t using the quadratic formula.

$$h = -16t^2 + 5t + 100$$
$$50 = -16t^2 + 5t + 100$$
$$0 = -16t^2 + 5t + 50$$
$$a = -16 \quad b = 5 \quad c = 50$$
$$t = \frac{-5 \pm \sqrt{5^2 - 4(-16)(50)}}{2(-16)}$$
$$t = \frac{-5 \pm \sqrt{3{,}225}}{-32}$$
$$t \approx -1.62 \text{ or } t \approx 1.93$$

Because time will not be negative, the elapsed time is approximately equal to 1.93 seconds.

Connections

Applications involving parabolas and quadratic equations can be found in a wide variety of occupations. An engineer may design a satellite dish in the shape of a parabola because its shape can reflect radio waves into a concentrated area. Manufacturers use quadratic equations to maximize revenue by adjusting price according to demand. Business owners use quadratic equations to optimize variables such as cost and profit.

Additional Examples

1. The height, in feet, of a soccer ball is given by the equation $h = -16t^2 + 45t$, where t is the time in seconds after the ball is kicked. What is the initial height and initial velocity of the soccer ball?

The initial height is the constant term. Because the equation could be written as $h = -16t^2 + 45t + 0$, the initial height is zero. The ball is kicked from ground level. You also could get this information by substituting zero for time t and by solving for the height h.
The initial velocity is the coefficient of the linear term. So, the initial velocity is 45 feet per second.

2. The height, in feet, of a soccer ball is given by the equation $h = -16t^2 + 45t$, where t is the time in seconds after the ball is kicked. How long is the soccer ball in the air before it hits the ground?

Substitute zero for h and solve by the factoring method.

$0 = -16t^2 + 45t$

$16t^2 - 45t = 0$

$t(16t - 45) = 0$

$t = 0$ or $16t - 45 = 0$

$t = 0$ or $t = \frac{45}{16} \approx 2.81$

The time $t = 0$ represents the time when the ball is first kicked, and the time $t \approx 2.81$ represents the time when the ball hits the ground.

The ball is in the air for approximately 2.81 seconds.