13.4 teacher notes

Objectives

- Determine the constant that makes a quadratic trinomial a perfect square.
- Write a perfect square trinomial as the square of a binomial.
- Solve quadratic equations by completing the square.

Vocabulary

Quadratic equation (Lesson 13-1)
Completing the square method
Perfect square trinomial
Factoring (Lesson 12-1)

Prerequisites

Squaring binomials

Factoring trinomials

Solving quadratic equations by evaluating square roots

Get Started

- Write $(x + 5)^2$ on the board. Ask students to square the binomial $x + 5$ by the FOIL Method. Write the simplified answer, $x^2 + 10x + 25$, next to the original expression.
- Repeat the process with $(x - 3)^2$ and $(x + 6)^2$. Their answers of $x^2 - 6x + 9$ and $x^2 + 12x + 36$, respectively, should also be written.
- Ask students to find a relationship between the coefficient of the middle term and the last term of each trinomial.
- After students find that the last term is the square of half the coefficient of the middle term, students should fill in the blanks of the following two equations:

$$(x + 10)^2 = x^2 + \underline{\qquad}x + 100$$
$$(x - 7)^2 = x^2 - 14x + \underline{\qquad}$$

- Students should check their answers by the FOIL Method.

Section 1

Expand Their Horizons

In Section 1, students will learn to complete the square for expressions of the form $x^2 + bx$. Completing the square will be used in solving quadratic equations which cannot be solved by factoring. They begin, however, by examining the perfect square trinomial.

A perfect square trinomial is the square of a binomial. The square of a binomial has one of the following two patterns:

$(a + b)^2 = a^2 + 2ab + b^2$

$(a - b)^2 = a^2 - 2ab + b^2$

Look at the resulting trinomials. In each trinomial, the first and the last terms are perfect squares. The first term in the trinomial is the square of the first term in the binomial, and the last term in the trinomial is the square of the last term in the binomial. The middle term in the trinomial is twice the product of the terms in the binomial. Knowledge of these binomial patterns allows one to square binomials without the use of the FOIL Method.

Square the binomial $x - 6$. The first term of the trinomial is the square of the first term of the binomial, so the first term in the trinomial is x^2. The last term in the trinomial is the square of the last term of the binomial, so the last term in the trinomial is $(-6)^2$ or 36. The middle term of the trinomial is twice the product of the terms in the binomial. The middle term of the trinomial is $(2)(x)(-6)$ or $-12x$. Therefore, $(x - 6)^2 = x^2 - 12x + 36$.

Once students understand the makeup of a perfect square trinomial, they are asked to complete the square. The method presented in the lesson only works when the leading coefficient of the quadratic is one. In other words, they are given an expression of the form $x^2 + bx$ (which is $1x^2 + bx$) and must find the last term of the perfect square trinomial. An analysis of a perfect square trinomial reveals that the last term is the square of half of the coefficient of the middle term. Consider the expression $x^2 - 12x +$ ____. To complete the square, the missing term is the square of half of -12. So, the missing term is $(-6)^2$ or 36.

A perfect squared trinomial can easily be factored. For $x^2 - 12x + 36$, the first term of each binomial factor is x, and the last term of each binomial factor is -6. So, $x^2 - 12x + 36$ factors into $(x - 6)^2$.

Common Error Alert

When factoring a perfect square trinomial whose leading coefficient is one, students may take the square root of the last term, rather than half of the coefficient of the middle term. This method is wrong because it will result in a correct answer only when the middle term is positive.

1 **Complete the square. $x^2 - 5x +$ ____** The last term of the perfect square trinomial is the square of half the coefficient of the middle term. The middle term is -5. Half of -5 is $-\frac{5}{2}$. To square a fraction, square the numerator and denominator. The square of -5 is 25. The square of two is four. Therefore, the last term of the expression is $\frac{25}{4}$.

It is common practice to keep the constant in terms of a fraction, rather than a decimal, when completing the square, because it is easier to take the square roots of fractions in subsequent steps.

2 **Factor: $x^2 - 5x + \frac{25}{4}$.** Every perfect square trinomial factors into the square of a binomial. The first term of the trinomial is the square of the first term of the binomial. So here, the first term in the binomial is x. The second term in the binomial is half the coefficient of the middle term in the trinomial. Therefore, the second term is $-\frac{5}{2}$. The trinomial factors into $\left(x - \frac{5}{2}\right)^2$. Encourage students to check their answers by multiplying.

The method of completing the square can be modeled with algebra tiles. For example, complete the square for $x^2 + 8x +$ ____. x^2 is modeled by one blue x-squared tile, and $8x$ is modeled by eight green x-tiles, such that half are to the right of the x-squared tile and half are below it. (See Figure 1.) The constant, which completes the square, is the number of 1's tiles required to make the figure a square. The missing area has a length and width of four, so its area is 4^2 or 16. (See Figure 2.) The perfect square trinomial is $x^2 + 8x + 16$.

Figure 1

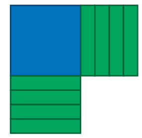
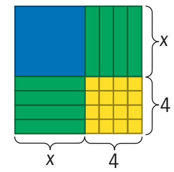


Figure 2



Additional Examples

1. Complete the square. $x^2 + 22x +$ ___

Divide 22 by two and square the result.

$22 \div 2 = 11$

$11^2 = 121$

The perfect square trinomial is $x^2 + 22x + 121$.

2. Factor: $x^2 + 3x + \frac{9}{4}$.

The first term of the squared binomial is x, and the last term is half of three or $\frac{3}{2}$.

$x^2 + 3x + \frac{9}{4} = \left(x + \frac{3}{2}\right)^2$

Section 2

Expand Their Horizons

In Section 2, completing the square will be used to solve quadratic equations. Completing the square will be used when the factoring or evaluating square roots methods cannot be used.

The first example in this section is $x^2 + 12x + 5 = 6$. Some students may attempt to subtract six from each side of the equation in an attempt to solve the problem by factoring. However, the resulting trinomial, $x^2 + 12x - 1$, is prime, so $x^2 + 12x - 1 = 0$ cannot be solved by factoring.

The first step is to isolate the two variable terms, x^2 and $12x$, on one side of the equation by subtracting five from each side of the equation. The resulting equation is $x^2 + 12x = 1$. Complete the square on the left hand side of the equation. $12 \div 2 = 6$ and $6^2 = 36$. To keep the equation balanced, 36 must be added to both the left and right hand side of the equation. The equation at this point is $x^2 + 12x + 36 = 37$. The trinomial is now a perfect square trinomial and can be factored. The resulting equation is $(x + 6)^2 = 37$ and can be solved by evaluating square roots. The quantity $x + 6$ equals $\sqrt{37}$ or $-\sqrt{37}$. By solving each equation, it is found that $x = -6 + \sqrt{37}$ or $x = -6 - \sqrt{37}$. The compound sentence $x = -6 + \sqrt{37}$ or $x = -6 - \sqrt{37}$ can also be written as $x = -6 \pm \sqrt{37}$. The solution set is $\{-6 + \sqrt{37}, -6 - \sqrt{37}\}$.

These values are real numbers although they are irrational. When the solution set consists of rational numbers, it means the equation could have been solved by the factoring method. Consider $x^2 - 2x + 5 = 20$. The first method below shows the equation being solved by completing the square. The second method shows the equation being solved by factoring.

Completing the Square

$$\begin{aligned} x^2 - 2x + 5 &= 20 \\ \underline{-5} \quad & \underline{-5} \\ x^2 - 2x \quad &= 15 \\ x^2 - 2x + \underline{\quad\quad} &= 15 + \underline{\quad\quad} \\ x^2 - 2x + 1 &= 15 + 1 \\ x^2 - 2x + 1 &= 16 \\ (x - 1)^2 &= 16 \end{aligned}$$

$x - 1 = \sqrt{16}$ or $x - 1 = -\sqrt{16}$

$x - 1 = 4$ or $x - 1 = -4$

$x = 5$ or $x = -3$

$\{5, -3\}$

Factoring

$$\begin{aligned} x^2 - 2x + 5 &= 20 \\ \underline{-20} \quad & \underline{-20} \\ x^2 - 2x - 15 &= 0 \\ (x - 5)(x + 3) &= 0 \end{aligned}$$

$x - 5 = 0$ or $x + 3 = 0$

$x = 5$ or $x = -3$

$\{5, -3\}$

In the second lesson example of Section 2, the leading coefficient on the left hand side of the equation is not one. The method presented for completing the square only works when the leading coefficient is one. Therefore, before completing the square, divide every term in the equation by the leading coefficient.

3 **Solve: $x^2 + 6x + 2 = -6$.** Start by subtracting two from each side of the equation to isolate the variable terms on one side. The equation becomes $x^2 + 6x = -8$. Because the leading coefficient is one, we can complete the square. Since $6 \div 2 = 3$ and $3^2 = 9$, add nine to each side of the equation. This makes the equation $x^2 + 6x + 9 = 1$. The trinomial factors into $(x + 3)^2$, so the equation is now $(x + 3)^2 = 1$. By evaluating square roots, $x + 3 = \sqrt{1}$, or $x + 3 = -\sqrt{1}$. Because $\sqrt{1} = 1$, $x + 3 = 1$ or $x + 3 = -1$. Solving each equation results in $x = -2$ or $x = -4$. The solution set is $\{-2, -4\}$.

These results are rational, so the equation could have been solving by factoring. The original equation was $x^2 + 6x + 2 + -6$. To make the right hand side of the equation equal to zero, add six to each side. The equation becomes $x^2 + 6x + 8 = 0$. The trinomial factors into $(x + 2)(x + 4)$. $x + 2 + 0$ or $x + 4 = 0$, so $x = -2$ or $x = -4$. The solution set is $\{-2, -4\}$.

Look Beyond

When the method of completing the square is applied to the standard form of a quadratic equation, $ax^2 + bx + c = 0$, the resulting solutions constitute the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. There are numerous uses for the quadratic formula. Finding missing dimensions in geometry and representing flight paths in physics are just two of them.

Additional Examples

1. Solve: $x^2 + 10x + 4 = 12$.

This problem cannot be solved by the factoring method because in the equation $x^2 + 10x - 8 = 0$, the trinomial is prime.

$$\begin{aligned} x^2 + 10x + 4 &= 12 \\ -4 \quad & \quad -4 \\ x^2 + 10x \quad &= 8 \end{aligned}$$

$$x^2 + 10x + \_\_\_\_ = 8 + \_\_\_\_$$
$$x^2 + 10x + 25 = 8 + 25$$
$$x^2 + 10x + 25 = 33$$
$$(x + 5)^2 = 33$$
$$x + 5 = \sqrt{33} \quad \text{or } x + 5 = -\sqrt{33}$$
$$x = -5 + \sqrt{33} \text{ or} \quad x = -5 - \sqrt{33}$$
$$\{-5 + \sqrt{33}, -5 - \sqrt{33}\}$$

2. Solve: $2x^2 - 9x + 6 = 10$.

In this example, the "±" symbol is used as soon as the square root is evaluated.

$$2x^2 - 9x + 6 = 10$$
$$\underline{-6} \quad \underline{-6}$$
$$2x^2 - 9x = 4$$
$$\frac{2x^2}{2} - \frac{9x}{2} = \frac{4}{2}$$
$$x^2 - \frac{9}{2}x = 2$$
$$x^2 - \frac{9}{2}x + \frac{81}{16} = 2 + \frac{81}{16}$$
$$x^2 - \frac{9}{2}x + \frac{81}{16} = \frac{32}{16} + \frac{81}{16}$$
$$\left(x - \frac{9}{4}\right)^2 = \frac{113}{16}$$
$$x - \frac{9}{4} = \pm\sqrt{\frac{113}{16}}$$
$$x = \frac{9}{4} \pm \frac{\sqrt{113}}{4}$$
$$x = \frac{9 \pm \sqrt{113}}{4}$$
$$\left\{\frac{9 + \sqrt{113}}{4}, \frac{9 - \sqrt{113}}{4}\right\}$$