13.3 teacher notes

Objective

- Solve quadratic equations by factoring.

Vocabulary

Quadratic equation (Lesson 13-1)
Factoring (Lesson 12-1)
Greatest common factor (Lesson 12-1)
Zero Product Property
Factor (Lesson 1-3)
Solution set (Lesson 5-1)
Binomial (Lesson 2-1)
Trinomial (Lesson 2-1)

Prerequisites

Factoring polynomials by removing the greatest common factor

Factoring trinomials by the *guess-and-check* method

Factoring trinomials by grouping

Solving linear equations in one variable

Rewriting equations in standard form

Get Started

- Divide the class into groups or pairs. Assign each group one of the examples below. In each, a number is "covered up." Tell the students to determine what number is "covered" for their example. Have a representative of each group share the group's answer with the class.
  $2 \cdot \square = 0$ $\square \cdot 5 = 0$ $1 \cdot \square = 0$ $-3 \cdot \square = 0$ $\square \cdot -4 = 0$
- The "covered" number in each example is zero. Encourage students to discuss why each covered number has to be zero. Students should conclude that the product of any number is zero. Lead the discussion until students arrive at this conclusion.
- Give all students the example $\square \cdot \Diamond = 0$. Ask students to determine what numbers are "covered" in this example. If necessary, lead the discussion so that students conclude that one of the "covered" numbers—and it doesn't matter which one—must be zero and that the other "covered" number can be any number (including zero).

- Tell students that this is the Zero Product Property, which will be studied in this lesson. Explain that the Zero Product Property states if a product equals zero, then at least one factor equals zero.

Section 1

Expand Their Horizons

In Section 1, students will solve quadratic equations by factoring. Factoring by removing the greatest common factor, factoring the difference of two squares, and factoring trinomials by creating a factor pair list, grouping, or guess-and-check are all methods that can be used in solving some quadratic equations. Recall that quadratic equations can have zero, one, or two real solutions. If the equation can be solved by factoring, the solution(s) will not only be real but will be rational numbers.

The Zero Product Property is used when solving quadratic equations by factoring. This property states that if the product of factors is equal to zero, then at least one of the factors must be equal to zero: If $ab = 0$, then $a = 0$ or $b = 0$. That is, the only way to get a product of zero is to multiply by zero. This rule extends for more than two factors. Factoring a polynomial is rewriting the polynomial as a product of simpler expressions. Thus, if a quadratic polynomial is equal to zero, then one of the simpler expressions into which it can be factored must be equal to zero.

The steps to solve quadratic equations by factoring are:

- Make one side equal to zero, rearranging the equation into standard form.
- Factor by removing the greatest common factor, as a difference of two squares, by using a factor pair list, by grouping, or with the guess-and-check method.
- Set each factor equal to zero, applying the Zero Product Property.
- Solve each linear equation by isolating the variable.

Consider the equation $x^2 - 14x + 48 = 0$. The equation is already in standard form. The factored form of $x^2 - 14x + 48 = 0$ is $(x - 6)(x - 8) = 0$. Now, the polynomial has been rewritten as the product of two factors, $(x - 6)$ and $(x - 8)$. According to the Zero Product Property, if a product is equal to zero, then one of the factors must be equal to zero. Each binomial factor is set equal to zero: $x - 6 = 0$ or $x - 8 = 0$. Solving both resulting linear equations will provide the two roots, or solutions, of the equation. The first, $x - 6 = 0$, is solved by adding six to both sides of the equation to get $x = 6$. The second, $x - 8 = 0$, is solved by adding eight to both sides of the equation to get $x = 8$. Most students will be able to solve linear equations such as these mentally. The roots of the quadratic equation $x^2 - 14x + 48 = 0$ are $x = 6$ and $x = 8$. The solution set is $\{6, 8\}$.

The solutions to quadratic equations can be checked; just as the solution to any equation can be checked. Check the solution set by substituting its values, one at a time, for the variable in the original equation. If the result is a true statement, the solution is correct. Stress to students checking one solution does not ensure that the other is correct.

To check the solutions found for the previous example, substitute one solution at a time into the original equation to determine if a true statement results. Substituting six for x in $x^2 - 14x + 48 = 0$, the equation becomes $(6)^2 - 14(6) + 48$. This expression simplifies to $36 - 84 + 48 = 0$ or $0 = 0$, which is a true statement. Thus, the root $x = 6$ checks. Substitute eight for x in the equation to get $(8)^2 - 14(8) + 48$, which simplifies to

$64 - 112 + 48 = 0$ or $0 = 0$. This is also a true statement. Thus, the root $x = 8$ also checks. Both solutions are correct.

Common Error Alert

Students may mistakenly believe if an equation cannot be factored that it has no solution. Factoring is only one method for solving quadratic equations, and it is applicable to equations having rational roots. Some quadratic equations cannot be solved by factoring. All quadratic equations can be solved by completing the square or by using the quadratic formula.

Next, consider the equation $y^2 + 49 + 14y = 0$. Although the polynomial is equal to zero, this equation is not in standard form. Standard form requires the variables be placed in descending order by exponent. The equation can be rewritten as $y^2 + 14y + 49 = 0$. Factoring the polynomial on the left side results in the equation $(y + 7)(y + 7) = 0$. Either $y + 7 = 0$ or $y + 7 = 0$. To solve these equations, seven is subtracted from both sides to get $y = -7$ or $y = -7$. The two solutions are the same rational number. Because the two solutions are the same number, the number needs to be listed only once in the solution set: $\{-7\}$. In cases such as this, $y = -7$ is called a multiple root. The equation $y^2 + 14y + 49 = 0$ is equivalent to the equation $(y - 7)(y - 7) = 0$. We say that $y = -7$ is a root of multiplicity two or that $y = -7$ is a double root.

Now, consider the equation $x^2 + 8x = 6x^2 - 4$. The first step in solving this quadratic equation is to make one side equal to zero. One way to accomplish this is to subtract $6x^2$ from both sides and then, add four to both sides to get $-5x^2 + 8x + 4 = 0$. Alternatively, we can subtract x^2 and $8x$ from both sides to get $0 = 5x^2 - 8x - 4$. Although both versions of the equation are correct, it is much easier to factor a trinomial when the leading coefficient is positive. For quadratic equations, this means that a positive x^2 term is more desirable. Using $0 = 5x^2 - 8x - 4$ and factoring yields $0 = (5x + 2)(x - 2)$. Set each factor equal to zero. The resulting equations are $x - 2 = 0$ or $5x + 2 = 0$. The first equation can be solved by adding two to both sides of the equation. The result is $x = 2$. The second equation, $5x + 2 = 0$, can be solved by subtracting two from each side to get $5x = -2$ and then, by dividing both sides by the coefficient five to get $x = -\frac{2}{5}$. Thus, the solution set for $x^2 + 8x = 6x^2 - 4$ is $\left\{-\frac{2}{5}, 2\right\}$.

Now, consider the quadratic equation $a^2 = 8a$. Subtracting $8a$ from both sides yields $a^2 - 8a = 0$. Recall the first type of factoring that should always be attempted is factoring out the greatest common factor. In this case the greatest common factor is a, which when factored out, yields $a(a - 8) = 0$. Therefore, the factors are a and $a - 8$. Set each factor equal to zero. Either $a = 0$ or $a - 8 = 0$. Subtract eight from both sides of the equation $a - 8 = 0$ to get $a = 8$. Therefore, the solution set for $a^2 = 8a$ is $\{0, 8\}$.

Consider the equation $6x^2 + 15x = 0$. It is already in standard form. The common factor $3x$ can be factored out. The resulting equation is $3x(2x + 5) = 0$. Set both factors equal to zero. The first equation is $3x = 0$, which can be solved by dividing both sides by three to get $x = 0$. The second equation is $2x + 5 = 0$, which can be solved by subtracting five from both sides and then, by dividing both sides by three to get $x = -2\frac{1}{2}$. The solution set for $6x^2 + 15x = 0$ is $\left\{0, -2\frac{1}{2}\right\}$.

Common Error Alert

Make sure that students solve equations of the form $ax = 0$ by dividing both sides of the equation by the coefficient a. In the previous example, $6x^2 + 15x = 0$, which factors into $3x(2x + 5) = 0$, students may incorrectly give $x = -3$ as the root by incorrectly solving $3x = 0$.

Consider $b^2 = 25$. First put the equation in standard form. The equation becomes $b^2 - 25 = 0$. The left side of the equation factors into the sum and difference of the same terms: $(b + 5)(b - 5) = 0$. Set each factor equal to zero and solve the two resulting equations. One equation, $b + 5 = 0$, becomes $b = -5$, and the other equation, $b - 5 = 0$, becomes $b = 5$. Thus, the solution set is $\{-5, 5\}$. Compare the preceding solution to the method used in Lesson 13-2, evaluating square roots. For the equation $b^2 = 25$, evaluate what numbers squared will equal 25. Because this simplifies to $b = \pm 5$, the result is clearly the same. Generally, evaluating square roots is preferred to factoring because evaluating square roots always results in the roots of the equation, even when those roots are not rational. But you can only use the Evaluating Square Roots Method on equations of the form: $ax^2 = k$ or $(x + a)^2 = k$.

1 **Solve: $(x - 8)(x + 2) = 0$.** The quadratic equation is already factored and has one side equal to zero. Applying the Zero Product Property, each factor is set equal to zero: $x - 8 = 0$ and $x + 2 = 0$. To solve the first equation, add eight to each side of the linear equation to get $x = 8$. To solve the second equation, subtract two from each side to get $x = -2$. The solution set is $\{8, -2\}$.

2 **Solve: $3x^2 + 7x - 10 = -4$.** The equation must first be put in standard form. That can be accomplished by adding four to both sides and combining the four with its like term of -10 on the left. The rewritten equation is $3x^2 + 7x - 6 = 0$. When the left side is factored, the resulting equation is $(3x - 2)(x + 3) = 0$. Set each factor equal to zero. The resulting equations are $3x - 2 = 0$ and $x + 3 = 0$. Solve the first equation by adding two to both sides to get $3x = 2$ and then, by dividing both sides by the coefficient three to get the solution $x = \frac{2}{3}$. Solve the second equation by subtracting three from both sides to get the solution $x = -3$. The solution set is $\left\{\frac{2}{3}, -3\right\}$.

3 **Solve: $2x^2 - 7x = 3x$.** The equation must first be put in standard form. This can be accomplished by subtracting $3x$ from both sides and combining the $-3x$ with its like term of $-7x$ on the left side. The rewritten equation is $2x^2 - 10x = 0$. The left side of the quadratic equation is factored by removing the greatest common factor, which is $2x$. The resulting equation is $2x(x - 5) = 0$. Set the common factor equal to zero. The resulting equation is $2x = 0$, which can be solved by dividing each side by the coefficient two to get $x = 0$. Set the remaining factor equal to zero. The resulting equation is $x - 5 = 0$, which can be solved by adding five to both sides to get $x = 5$. The solution set is $\{0, 5\}$.

Look Beyond

Factoring is only one method that can be used to solve quadratic equations, and it is applicable only to factorable equations. Other methods such as completing the square (Lesson 13-4) and the quadratic formula (Lesson 13-5) can find the solutions to any quadratic equation. Many people consider factoring to be the quicker method when it can be used.

Completing the square actually uses evaluating square roots from Lesson 13-2 and factoring, and completing the square is used to derive the quadratic formula.

Additional Examples

1. Solve:
$\boldsymbol{2m^2 + 5m - 2 = 3m^2 - 7m + 34.}$

First, the equation must be put in standard form. The best choice is to rewrite the equation so that the leading coefficient is positive. Subtract $2m^2$ from each side to get $5m - 2 = m^2 - 7m + 34$. Then, subtract $5m$ from each side to get $-2 = m^2 - 12m + 34$. Finally, add two to each side to get $0 = m^2 - 12m + 36$. Then, factor the equation. Its factored form is $0 = (m - 6)(m - 6)$. $0 = m - 6$ or $0 = m - 6$. Add six to both sides. $6 = m$ or $6 = m$. The solution is $m = 6$ with six being a root of multiplicity two or a double root. The solution set is $\{6\}$.

2. Solve: $\boldsymbol{y^2 - 81 = 0.}$

This equation is already in standard form. The left side of the equation is the difference of two squares and is factored as a product of conjugates. Factoring the left side gives us $(y + 9)(y - 9) = 0$. Set the first factor equal to zero, $y + 9 = 0$ and subtract nine from both sides. One solution is $y = -9$. Set the second factor equal to zero, $y - 9 = 0$ and add nine to both sides. The other solution is $y = 9$. The solution set is $\{-9, 9\}$.