13.2

teacher notes

Objective

- Solve quadratic equations of the form $ax^2 = k$ by evaluating square roots.
- Solve quadratic equations of the form $(x + a)^2 = k$ by evaluating square roots.

Prerequisites

Solving linear equations

Simplifying square roots of perfect square integers

Expressing square roots of integers that are not perfect squares

Writing and understanding disjunctions

Using set notation

Vocabulary

Solution (Lesson 3-2)
Quadratic equation (Lesson 13-1)
Root
Square root (Lesson 1-4)
Real number (Lesson 1-1)
Empty set (Lesson 1-1)

Get Started

- Divide the students in the class into two groups. Write the numerals –3, –2, and –1 on the left side of the board; students in one group will use these numbers. Write the numerals 1, 2, and 3 on the right side of the board; students in the other group will use these numbers.
- Ask students to choose an integer from the set assigned to their group and write it down. Then, ask them to determine the square of the number they chose and to write it down.
- Say, “Raise your hand if the *square* of your number is four.” Call on those students to tell their number. Say, “Some students chose two and wrote four as the square of it, and some students chose negative two and wrote four as the square of it.”

- Say, "We have just shown that there are two numbers with a square of four. Write the equation $x^2 = 4$ on the board. Ask students to solve the equation by saying what values of x make it true. They should be able to say that the solutions are two and negative two.
- Remind the class that $x^2 = 4$ is a quadratic equation. Rewrite it in standard form $x^2 - 4 = 0$.
- Repeat the above for the squares nine and one.
- Ask students to solve the equation $x^2 = 0$ by asking what value of x can be squared to get zero? Discuss the fact that this equation has only one solution because zero is the only number whose square is zero. There is no difference between $+0$ and -0.
- Ask if there is a number that can be squared to get -4. Discuss the fact that the equation $x^2 = -4$ has no *real* number solution, and that $x^2 =$ (*any negative number*) has no real number solution.
- Say, "Today we will learn to solve certain types of quadratic equations, and we will see that a quadratic equation can have as many as two real solutions, but sometimes has just one real solution, and sometimes has no real solution."

Section 1

Expand Their Horizons

This lesson will introduce one method for solving quadratic equations. In Section 1, the quadratic equations will be in the form $ax^2 = k$ or $ax^2 + b = k$, where a, b, and k are constants.

If $a = 1$, then the equation $ax^2 = k$ is $x^2 = k$, and if k is positive, the solutions are $\sqrt{k}$ and $-\sqrt{k}$. The expression $\sqrt{k}$ indicates the *principal square root* of k, which is always nonnegative. The expression $-\sqrt{k}$ indicates the *opposite* of the principal square root of k, also called the *negative* square root of k . For example, if k is 16, the equation is $x^2 = 16$. The solutions are $\sqrt{16}$ and $-\sqrt{16}$, which are four and negative four in simplest form.

If k is negative, there is no real solution because there is no real number whose square is negative. For example, the equation $x^2 = -16$ has no real solution since no value of x produces -16 when squared.

If k is zero, the equation is $x^2 = 0$, and zero is the only value of x that makes the equation true. The only solution to the equation is zero.

There are three ways commonly used for expressing the solutions to a quadratic equation. One way is to write a *disjunction*. For the equation $x^2 = 9$, the solutions can be expressed by writing the disjunction "$x = 3$ or $x = -3$." Another way uses the symbol "$\pm$." So, the solutions to the same equation could be expressed by writing "$x = \pm 3$," which should be read as "x is equal to positive three or x is equal to negative three." The shortcut way to read this is "x is equal to plus or minus three." The third way to express the solutions is to write the solution set. The solution set for $x^2 = 9$ is $\{3, -3\}$.

Common Error Alert

Some incorrect ways to express the solutions to the equation $x^2 = 9$ are shown below, along with explanations and the corresponding correct ways.

Incorrect

$x = 3$ and $x = -3$

Explanation

x cannot be two different numbers at the same time.

Correct

$x = 3$ or $x = -3$

Incorrect

The solutions are 3 or –3.

Explanation

The word "or" suggests that 3 and –3 are alternative solutions. In fact, they are both solutions.

Correct

The solutions are 3 and –3.

Incorrect

The solution is {3, –3}.

Explanation

{ } is the symbol for a set.

Correct

The solution set is {3, –3}.

If an equation is not in the form $x^2 = k$, the quadratic term (the term with the exponent two) must first be isolated on one side of the equation to get the equation in this form. For example, to solve the equation $4x^2 - 12 = 4$, isolate x^2 on the left side of the equation by first adding 12 to get $4x^2 = 16$ and then, by dividing by four to get $x^2 = 4$.

1 **Solve: $3x^2 - 10 = 65$.** Begin by isolating x^2. Add 10 to both sides and then, divide both sides by three. The equation is now $x^2 = 25$. To solve, write the related disjunction $x = \sqrt{25}$ or $x = -\sqrt{25}$. Simplify each radical expression to get $x = 5$ or $x = -5$. Express the answers in set notation as {5, –5}.

Common Error Alert

Students with prior algebra experience may insist on "taking the square root of both sides" to solve equations of the type $x^2 = k$. Although on the surface it appears to produce identical results to the method of evaluating square roots, explain to students that $\sqrt{x^2} = x$ only when x is a non-negative number, and the correct simplification is $\sqrt{x^2} = |x|$.

Additional Examples

1. Solve: $-3x^2 - 4 = -151$.

Isolate x^2 by adding four to both sides of the equation, then dividing both sides by negative three. When the equation has the form $x^2 = k$, write the disjunction and simplify the radical expressions.

$$-3x^2 - 4 = -151$$
$$+4 \qquad +4$$
$$-3x^2 = -147$$
$$\frac{-3x^2}{-3} = \frac{-147}{-3}$$
$$x^2 = 49$$
$$x = \sqrt{49} \text{ or } x = -\sqrt{49}$$
$$x = 7 \quad \text{ or } x = -7$$

Solution set: {7, –7}

2. Solve: $6 + 2x^2 = 40$.

In this case, the solutions are radical expressions that cannot be simplified.

$$6 + 2x^2 = 40$$
$$-6 \qquad\quad -6$$
$$2x^2 = 34$$
$$\frac{2x^2}{2} = \frac{34}{2}$$
$$x^2 = 17$$
$$x = \sqrt{17} \text{ or } x = -\sqrt{17}$$

Solution set: $\{\sqrt{17}, -\sqrt{17}\}$

Section 2

Expand Their Horizons

In Section 2, the quadratic equations will be in the form $a(x + b)^2 = k$ or $a(x + b)^2 + c = k$, where a, b, c, and k are constants. To solve an equation of either type, first isolate the expression $(x + b)^2$. Then, write the appropriate disjunction and solve the two equations that appear in the disjunction.

The first equation in the lesson is $(x + 4)^2 = 9$, and $(x + 4)^2$ is already isolated. (Note that this is in the form $a(x + b)^2 = k$, with $a = 1$, $b = 4$, and $k = 9$.) The appropriate disjunction for this equation is $x + 4 = 3$ or $x + 4 = -3$. Solving these two equations provides the solutions to the original equation. The solutions are negative one and negative seven.

The second equation in the lesson is $3(x - 1)^2 - 1 = 38$, which is in the form $a(x + b)^2 + c = k$, with $a = 3$, $b = -1$, $c = -1$, and $k = 38$. For this equation, it is necessary to isolate $(x - 1)^2$. First, add one to each side to get $3(x - 1)^2 = 39$ and then, divide each side by three to get $(x - 1)^2 = 13$. Next, write the disjunction $x - 1 = \sqrt{13}$ or $x - 1 = -\sqrt{13}$. Solve these two equations to get $x = 1 + \sqrt{13}$ or $x = 1 - \sqrt{13}$. The solutions are $1 + \sqrt{13}$ and $1 - \sqrt{13}$. Note that when a solution is the sum or difference of a rational number and an irrational number, it is conventional to write the rational number first. So, the conventional way to write the solutions is $1 + \sqrt{13}$ and $1 - \sqrt{13}$. However, it is also correct to write $\sqrt{13} + 1$ and $-\sqrt{13} + 1$.

2 **Solve: $2(x + 2)^2 + 25 = 25$.** Isolate the expression $(x + 2)^2$ by subtracting 25 from both sides and then, by dividing both sides by two. The equivalent equation that results is $(x + 2)^2 = 0$. The disjunction for this equation is $x + 2 = 0$ or $x + 2 = -0$. Because zero and negative zero represent the same number, it is evident that there is no need for a disjunction (because the opposite of zero is zero). Solve $x + 2 = 0$ by subtracting two from both sides to get $x = -2$. So, this quadratic equation has only one solution, negative two. The solution set is $\{-2\}$.

3 **Solve: $(x - 5)^2 = -3$.** Since the equation shows the square of an expression equal to a negative number, there is no real solution.

Common Error Alert

For the equation $(x - 5)^2 = -3$, students may write the disjunction $x - 5 = -\sqrt{3}$ or $x - 5 = \sqrt{3}$. Remind them a disjunction should only be written when the equation shows the square of an expression equal to a *positive* number.

Look Beyond

A quadratic equation of the form $x^2 = k$ or $(x + b)^2 = k$, where $k < 0$, has *no real solution*. In future courses, students will learn that while such an equation has no *real* solution, it does have two *imaginary* solutions. An imaginary number is the square root of a negative number. The imaginary unit is $\sqrt{-1}$, represented by the symbol i, and it is a factor of any imaginary number. For example, the equation $x^2 = -25$ has the two imaginary solutions $5i$ and $-5i$, as shown below.

$$x^2 = -25$$
$$x = \sqrt{-25} \text{ or } x = -\sqrt{-25}$$
$$x = \sqrt{-1} \cdot \sqrt{25} \text{ or } x = -\sqrt{-1} \cdot \sqrt{25}$$
$$x = i \cdot 5 \text{ or } x = -i \cdot 5$$
$$x = 5i \text{ or } x = -5i$$

Additional Examples

1. Solve: $\frac{1}{2}(x - 4)^2 + 3 = 5$.

Subtract three from both sides of the equation and then, multiply both sides by the reciprocal of $\frac{1}{2}$, which is two. Then, write the appropriate disjunction and solve the equations in the disjunction.

$$\frac{1}{2}(x-4)^2 + 3 = 5$$
$$-3 \quad -3$$
$$\frac{1}{2}(x-4)^2 = 2$$
$$2 \cdot \frac{1}{2}(x-4)^2 = 2 \cdot 2$$
$$(x-4)^2 = 4$$
$$x - 4 = \sqrt{4} \text{ or } x - 4 = -\sqrt{4}$$
$$x - 4 = 2 \quad \text{or } x - 4 = -2$$
$$+4 + 4 \qquad +4 \quad +4$$
$$x = 6 \quad \text{or} \quad x = 2$$

Solution set: {2, 6}

2. Solve: $5 - 2(3 + x)^2 = -45$.

Subtract five from both sides of the equation and then, divide both sides by negative two. Then, write the appropriate disjunction and solve the equations in the disjunction.

$$5 - 2(3 + x)^2 = -45$$
$$-5 \qquad\qquad -5$$
$$-2(3 + x)^2 = -50$$
$$\frac{-2(3 + x)^2}{-2} = \frac{-50}{-2}$$
$$(3 + x)^2 = 25$$
$$3 + x = \sqrt{25} \text{ or } 3 + x = -\sqrt{25}$$
$$3 + x = 5 \quad \text{or } 3 + x = -5$$
$$-3 \quad -3 \qquad -3 \quad -3$$
$$x = 2 \quad \text{or} \quad x = -8$$

Solution set: {–8, 2}

Calculator Problem

In this activity, students will solve quadratic equations by using a TI-83 graphing calculator.

Example 1: Solve the equation $3x^2 - 10 = 65$ graphically.

Begin by writing the equation in an equivalent form with zero on one side of the equals sign. The equation $3x^2 - 10 = 65$ is rewritten in the equivalent form $3x^2 - 75 = 0$. Next, write an associated function for the equation, using the expression on the nonzero side of the equation to define the function. For the equation $3x^2 - 75 = 0$, the associated function is $y = 3x^2 - 75$. If the function $y = 3x^2 - 75$ is graphed, the x-intercepts of the graph are the solutions to $3x^2 - 75 = 0$ because the x-intercepts are the x-values that make $y = 0$. The x-intercepts of the graph of the function are called the zeros of the function. Students can find the zeroes using the **zero** function in the calculator.

To solve the equation $3x^2 - 75 = 0$ by using a graphing calculator, do the following:

1. Enter the quadratic expression $3x^2 - 75$ into the calculator using the [Y=] button. **(Figure 1)**
2. Graph the function: Press [ZOOM] [6] to use the standard window. This window does not show enough of the graph. Press [WINDOW] and use [▼] to set Ymin = –80 and Ymax = 40. Then, press [GRAPH].
3. Select [CALC] by pressing [2nd] [TRACE].
4. Select **2:zero** by pressing [2]. The current graph is displayed with **LeftBound?** in the bottom-left corner. **(Figure 2)**

5. Press [◄] or [►] to move the cursor to the left of an x-intercept. Press [ENTER]. Now **RightBound?** is displayed. Press [►] to move the cursor to the right of the x-intercept. Press [ENTER]. Now **Guess?** is displayed. Press [ENTER]. The calculator now displays

Zero
X = –5 Y = 0

in the bottom-left corner. **(Figure 3)**

6. Repeat Steps 3–5 to find the other solution. The calculator now displays

Zero
X = –5 Y = 0

in the bottom-left corner.

The zeroes of the function $y = 3x^2 - 75$ are five and negative five. Therefore, $x = 5$ and $x = -5$ are the solutions of the equation $3x^2 - 10 = 65$.

Figure 1

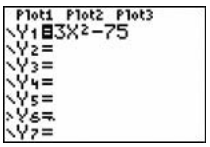


Figure 2

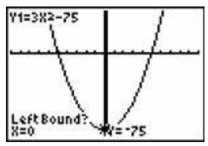
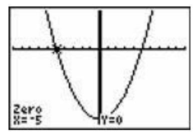


Figure 3



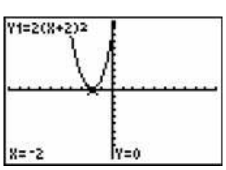
Example 2: Solve the equation $3(x - 1)^2 - 1 = 38$ graphically and compare the graphic solutions to the algebraic solutions.

To solve $3(x - 1)^2 - 1 = 38$ with the graphing calculator, remember to use the equivalent equation $3(x - 1)^2 - 39 = 0$ and to enter the related equation $y = 3(x - 1)^2 - 39$ into the graphing calculator. Repeat the procedure described in Example 1. Enter the quadratic expression $3(x - 1)^2 - 39$ using the [Y=] button. The same viewing window that was used in Example 1 will adequately display the function (Ymin = –80 and Ymax = 40). The zeros are –2.605551 and 4.6055513. In the lesson, the algebraic solutions to $3(x - 1)^2 - 1 = 38$ were given as $1 + \sqrt{13}$ and $1 - \sqrt{13}$. To compare the graphic solutions and the algebraic solutions, find decimal approximations for $1 + \sqrt{13}$ and $1 - \sqrt{13}$. Exit the GRAPH mode by pressing [2nd] [QUIT]. Press [1] [+] [2nd] [x^2] [1] [3] [ENTER] to approximate $1 + \sqrt{13}$. The result is 4.605551275, which is a slightly better approximation for the exact solution $1 + \sqrt{13}$ than is 4.6055513. Press [1] [−] [2nd] [x^2] [1] [3] [ENTER] to approximate $1 - \sqrt{13}$. The result is –2.605551275, which is a slightly better approximation for the exact solution $1 - \sqrt{13}$ than is –2.605551.

Example 3: Solve the equation $2(x + 2)^2 + 25 = 25$ graphically and compare the graphic solutions to the algebraic solutions.

Use the equivalent equation $2(x + 2)^2 = 0$. Using the calculator's **zero** function to solve the equation produces an error. However, **TRACE** can be used to verify that the algebraic solution in the lesson, $x = -2$, is correct. Enter the quadratic expression $2(x + 2)^2$ using the [Y=] button. Press [ZOOM] [6] to use the standard window. Press [TRACE] [(−)] [2] [ENTER]. On the bottom of the screen **X = –2 Y = 0** appears. **(Figure 4)**

Figure 4



Example 4: Solve the equation $(x - 5)^2 = -3$ graphically, if possible.

Use the equivalent equation $(x - 5)^2 + 3 = 0$. Enter the quadratic expression $(x - 5)^2 + 3$ using the Y= button. Press ZOOM 6 to use the standard window. Because the graph **(Figure 5)** does not intersect the *x*-axis, there is no solution. The solution set is the empty set ∅.

Figure 5