12.6 teacher notes

Objective

- Factor polynomials which require more than one method of factoring.

Vocabulary

Conjugates (Lesson 11-5)
Difference of two squares (Lesson 11-5)
Factoring (Lesson 12-1)
Greatest common factor (Lesson 12-1)

Prerequisites

Factoring polynomials by removing the greatest common factor

Factoring the difference of two squares

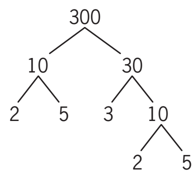
Factoring trinomials of the form $ax^2 + bx + c$

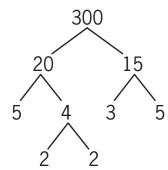
Factoring by grouping

Get Started

- Review how to find the prime factorization of a whole number with the class. Say, "One method for finding the prime factorization is to use a factor tree. In a factor tree, two branches are drawn from the number to each factor in one of its factor pairs. Then, branches continue to be drawn from each of those factors, until the end of each branch shows a prime number."
- With the students, draw a factor tree on the board to show the prime factorization of 300. Start with $10 \cdot 30$. Note that although $10 \cdot 30$ is a factorization of 300, it is not the *prime* factorization of 300 because neither 10 nor 30 is prime. Continue factoring drawing more branches of the tree. Ask the students to determine when a branch is finished. When a prime number is at the end of a branch, the branch ends.
- Draw a second factor tree for 300, starting with a different pair of factors, for example $20 \cdot 15$. Continue the tree until the prime factorization is reached.

- Rearrange the factors so that the factors are in increasing order. Show that the resulting prime factorization is $2 \cdot 2 \cdot 3 \cdot 5 \cdot 5$ for both trees.

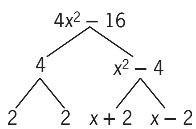


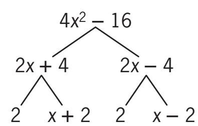


$$300 = 2 \cdot 5 \cdot 3 \cdot 2 \cdot 5$$
$$= 2 \cdot 2 \cdot 3 \cdot 5 \cdot 5$$

$$300 = 5 \cdot 2 \cdot 2 \cdot 3 \cdot 5$$
$$= 2 \cdot 2 \cdot 3 \cdot 5 \cdot 5$$

- Say, "Today, we will explore how to find the prime factorization of a polynomial. Sometimes, when a polynomial is factored, we can continue to factor one or both of the factors. Just like a factor tree for a whole number can have many rows of branches, the factor tree for a polynomial can sometimes require several steps in order to find the *complete*, or *prime* factorization. We will know when the factorization is complete when each factor is a monomial or a prime polynomial."
- Make a factor tree for the polynomial $4x^2 - 16$ with the class. Have the class consider that both of the factor trees below ultimately lead to the same complete factorization, $4(x + 2)(x - 2)$.





Prime factorization:
$$4x^2 - 16 = 2 \cdot (x + 2) \cdot 2 \cdot (x - 2)$$
$$= 2 \cdot 2 \cdot (x + 2) \cdot (x - 2)$$
$$= 4(x + 2)(x - 2)$$

Prime factorization:
$$4x^2 - 16 = 2 \cdot (x + 2) \cdot 2 \cdot (x - 2)$$
$$= 4(x + 2)(x - 2)$$

Section 1

Expand Their Horizons

In this lesson, students will combine the factorization methods learned throughout this module to completely factor polynomials. To completely factor each polynomial in this lesson, it is necessary to use more than one method or more than one application of the same method.

Before starting the lesson, review the methods students have learned for factoring polynomials. As review, students should factor one polynomial for each method of factorization:

Factoring by removing the greatest common factor: $3x^2 + 9x = 3x(x + 3)$
Factoring by grouping: $2x^3 + 8x^2 + 5x + 20 = 2x^2(x + 4) + 5(x + 4) = (x + 4)(2x^2 + 5)$
Factoring the difference of two squares: $100x^2 - 9 = (10x + 3)(10x - 3)$
Factoring $x^2 + bx + c$: $x^2 - 8x - 20 = (x - 10)(x + 2)$
Factoring $ax^2 + bx + c$: $6x^2 - 5x - 4 = (3x - 4)(2x + 1)$

Finding the complete factorization of a polynomial is similar to finding the prime factorization of a whole number. When finding the prime factorization of a whole number, the number is first expressed as the product of two factors. Then, the resulting two factors are examined to determine whether they are *prime* (having only one and themselves as factors) or *composite* (having factors other than one and themselves). If a factor is composite, it is factored, and the process repeats until each factor is prime. The prime factorization is expressed as the product of all the prime factors.

When completely factoring a polynomial, the same process is used. A factor of a polynomial is considered to be a *prime factor* if it is a monomial or it is a prime polynomial. A *prime polynomial* is one whose only factors are one and itself. For example, the binomial $x - 4$ is a *prime binomial*, since it has no factors other than 1 and $x - 4$.

Some of the expressions factored in Lesson 12-3 required more than one application of a factorization method. When factoring $x^4 - 16$, the factorization of the difference of two squares was used twice. The factorization $(x^2 + 4)(x^2 - 4)$ showed an incomplete factorization, and the factor $x^2 - 4$ was further factored, making the complete factorization $(x^2 + 4)(x + 2)(x - 2)$.

When completely factoring a polynomial, it is most efficient to remove the common factor (if any) in the first step. This is not required when the common factor does not include a variable, but doing so usually makes the remaining expression easier to factor. If a variable is part of the common factor, factoring it out first often will make the remaining expression a quadratic, which can then be factored using the other methods in this module.

Students may not see the entire common factor in an expression. For example in the expression $8b^5 - 2b^3$, the 2 may be factored out, resulting in $2(4b^5 - b^3)$. Each factor must be examined to determine if there is another common factor that can be removed. It may be more efficient to factor out the greatest common factor, but it is not wrong to factor out common factors in stages so long as the final factorization is complete.

1 **Factor: $8b^5 + 2b^3$.** The terms of the binomial $8b^5 - 2b^3$ have a common factor of $2b^3$. Remove the common factor to get $2b^3(4b^2 - 1)$. The monomial factor $2b^3$ is prime; the factor $4b^2 - 1$ is the difference of two squares, and factors into the product $(2b + 1)(2b - 1)$. The complete factorization of $8b^5 - 2b^3$ is $2b^3(2b + 1)(2b - 1)$.

Common Error Alert

Students often abandon prime factors as they concentrate on factoring composite factors. For example, they may omit the factor $2b^3$ and write the final factorization as $(2b + 1)(2b - 1)$. Encourage students to develop a habit of re-copying each factor onto the next line of his or her paper when finding factorizations, replacing composite factors with their factorizations. Visual learners may prefer to circle prime factors before they go on to factor composite factors. Writing the prime factorization requires that they write the product of all the circled expressions.

2 **Factor: $4rs^2 - 16rs - 48r$.** Removing the greatest common factor, $4r$, leaves a trinomial in s of the general form $x^2 + bx + c$. To factor $s^2 - 4s - 12$, look for two integers with a product of –12 and a sum of –4. The complete factorization is $4r(s + 2)(s - 6)$. Although the monomial factor $4r$ could be written as the product of prime factors $2 \cdot 2 \cdot r$, monomial factors are usually expressed as a single product.

3 **Factor: $2p^5 + 7p^4 + 3p^3$.** In this expression, the greatest common factor is p^3. When the common factor is removed, an expression in p of the general form $ax^2 + bx + c$ remains to be factored. The trinomial $2p^2 + 7p + 3$ factors into the product $(2p + 1)(p + 3)$. The complete factorization of $2p^5 + 7p^4 + 3p^3$ is $p^3(2p + 1)(p + 3)$.

4 **Factor: $t^3 + 3t^2 - t - 3$.** This expression is a 4-term polynomial in t. After confirming that the four terms have no common factor, students should turn to factoring by grouping (their only other strategy for factoring 4-terms). If the second and third terms of the polynomial are interchanged using the Commutative Property of Addition, the polynomial is $t^3 - t + 3t^2 - 3$, which can be factored using the grouping strategy: $t^3 - t + 3t^2 - 3 = t(t^2 - 1) + 3(t^2 - 1) = (t^2 - 1)(t + 3) = (t + 1)(t - 1)(t + 3)$. Some students may not swap the second and third terms but leave the polynomial as written as factor using a different grouping strategy: $t^3 + 3t^2 - t - 3 = t^2(t + 3) - (t + 3) = (t + 3)(t^2 - 1) = (t + 3)(t + 1)(t - 1)$. Show both methods to the class, emphasizing the idea that all valid methods lead to the same complete factorization of a polynomial. $(t + 1)(t - 1)(t + 3)$ and $(t + 3)(t + 1)(t - 1)$ are equivalent by the Commutative Property of Multiplication.

5 **Factor: $a^4 - b^4$.** This binomial shows the difference of two squares and factors as the product $(a^2 + b^2)(a^2 - b^2)$. The second factor shows the difference of two squares and factors as $(a + b)(a - b)$. The complete factorization of $a^4 - b^4$ is $(a^2 + b^2)(a + b)(a - b)$.

Look Beyond

Now that students are familiar with the basic methods for factoring polynomials, they are ready to apply the techniques. In the next module, they will be exposed to one of the most useful applications of factoring—solving equations. The *Zero Product Property* states that if $ab = 0$ (where a and b are any expressions), then at least one of the statements $a = 0$ and $b = 0$ must be true. In other words, if two factors are shown to have a product of 0, it must be true that at least one of the factors is zero.

This property can be used to solve linear equations like $3x + 12 = 0$. If the left side of the equation is factored, the equation is equivalent to $3(x + 4) = 0$, and it must be true that either three is equal to zero, or $x + 4$ is equal to zero. Since three is never equal to zero and $x + 4 = 0$ only when $x = -4$, the solution is $x = -4$.

Additional Examples

1. Factor completely: $25x^4 - 100x^2$.

Remove the greatest common factor, $25x^2$, first, and then factor the difference of two squares.

$25x^4 - 100x^2$

$25x^2(x^2 - 4)$

$25x^2(x + 2)(x - 2)$

2. Factor completely: $8m^4 + 20m^3 - 12m^2$.

Remove the greatest common factor, $4m^2$, first, and then factor the trinomial in m of the general form $ax^2 + bx + c$.

$8m^4 + 20m^3 - 12m^2$

$4m^2(2m^2 + 5m - 3)$

$4m^2(2m - 1)(m + 3)$