12.5

teacher notes

Objectives

- Factor trinomials of the form $ax^2 + bx + c$, where a $\neq$ 0, 1, using the *guess-and-check* method.
- Factor trinomials of the form $ax^2 + bx + c$, where a $\neq$ 0, 1, using the *grouping* method.

Vocabulary

Factor (Lesson 1-3)
Polynomial (Lesson 2-1)
Factoring (Lesson 12-1)
Trinomial (Lesson 2-1)
Coefficient (Lesson 2-1)
Binomial (Lesson 2-1)
FOIL Method (Lesson 11-4)
Like terms (Lesson 2-4)
Greatest common factor (Lesson 12-1)

Prerequisites

Multiplying binomials by the FOIL Method

Factoring by grouping

Factoring trinomials of the form $x^2 + bx + c$

Combining like terms

Get Started

- Review the guess-and-check method that was used in Lesson 12-4 to factor a trinomial of the form $x^2 + bx + c$. Use any trinomial of this type, for instance $x^2 + 10x + 24$, as an example. Ask the students how to factor this trinomial using the guess-and-check method. List the possible factors and ask the students to multiply each set of factors until they have found the correct factorization.

Possible Factors	Product
$(x + 24)(x + 1)$	$x^2 + x + 24x + 24 = x^2 + 25x + 24$
$(x + 12)(x + 2)$	$x^2 + 2x + 12x + 24 = x^2 + 14x + 24$
$\mathbf{(x + 6)(x + 4)}$	$\mathbf{x^2 + 4x + 6x + 24 = x^2 + 10x + 24}$

  Point out to the students that the constant term in each product is 24, but that the coefficient of the middle term varies depending on which factors of 24 were used. It is also important to point out that the middle term is the sum of the Outer and Inner products when using the FOIL Method because these individual products will be referred to in the lesson.

- Ask the students what would happen if the coefficient of the x term in the binomials were not equal to one.

- Divide the class into groups or pairs. Assign each group one of the expressions below.
- Ask each group to use the FOIL Method to rewrite the expression assigned to them. Have a representative of each group share the group's answer with the class.

Expression	**Product**
$(6x + 5)(x + 2)$	$6x^2 + 12x + 5x + 10 = 6x^2 + 17x + 10$
$(3x + 5)(2x + 2)$	$6x^2 + 6x + 10x + 10 = 6x^2 + 16x + 10$
$(3x + 2)(2x + 5)$	$6x^2 + 15x + 4x + 10 = 6x^2 + 19x + 10$
$(6x + 1)(x + 10)$	$6x^2 + 60x + x + 10 = 6x^2 + 61x + 10$
$(3x + 10)(2x + 1)$	$6x^2 + 3x + 20x + 10 = 6x^2 + 23x + 10$
$(3x + 1)(2x + 10)$	$6x^2 + 30x + 2x + 10 = 6x^2 + 32x + 10$

- Note that all of the trinomial products have a leading coefficient of 6 and a final constant of 10. Also, explain to students that listing the factor pairs of 6: (1)(6) and (2)(3), and 10: (1)(10) and (2)(5), would not be the most efficient way to determine the correct factorization of this, or a similar, trinomial. Additional techniques can be used to factor quadratic trinomials with leading coefficients other than one.

Section 1

Expand Their Horizons

In Section 1, students will factor quadratic trinomials with a leading coefficient other than one; that is, trinomials of the form $ax^2 + bx + c$, where $a \neq 0$ and $a \neq 1$. In this section they will use the guess-and-check method, also sometimes called the trial-and-error method.

Common Error Alert

Students may be reluctant to check their results. In Lesson 12-4, checking was used to verify that an answer was correct. However, the guess-and-check method uses *guesses*, and these guesses *must* be checked to determine the correct solution. In fact, multiplying the binomials should be considered part of the process for determining the factorization and not merely a check of one's work.

The guess-and-check method uses factor pairs of the first and last terms of the trinomial to determine the terms of the binomial factors. However, as pointed out in the "Get Started" exercise, there are often several possible combinations of these factors. Each combination must be checked to determine which yields the correct middle term.

Consider the quadratic trinomial $4x^2 + 16x + 15$. The factors of the first term could be either x and $4x$ or $2x$ and $2x$.

$$(x \quad)(4x \quad)$$
$$(2x \quad)(2x \quad)$$

Note that the variable portion of the term, x^2, is divided "evenly" into an x in each factor. This should always be the case; doing so will create like linear (x) terms in the middle terms of the polynomial. The factors of the last term could be either 1 and 15 or 3 and 5.

$$(\quad 1)(\quad 15)$$
$$(\quad 3)(\quad 5)$$

The order of the factors of the last term can make a difference when multiplying the binomials, so all possible combinations should be listed.

Note that changing the order of only the first factors or only the second factors is necessary, but changing the order of both will lead to duplicates. For example, $(x + 3)(4x + 5)$ is the same as $(4x + 5)(x + 3)$. Each possible combination of first-term factors and last-term factors are placed in binomials, and the binomials are multiplied using the FOIL Method. The binomial pair that produces the original expression is the correct factorization.

Example: $4x^2 + 16x + 15$

Trial Factors	Product
$(x + 1)(4x + 15)$	$4x^2 + 15x + 4x + 15 =$ $4x^2 + 19x + 15$
$(x + 15)(4x + 1)$	$4x^2 + x + 60x + 15 =$ $4x^2 + 61x + 15$
$(x + 3)(4x + 5)$	$4x^2 + 5x + 12x + 15 =$ $4x^2 + 17x + 15$
$(x + 5)(4x + 3)$	$4x^2 + 3x + 20x + 15 =$ $4x^2 + 23x + 15$
$(2x + 1)(2x + 15)$	$4x^2 + 30x + 2x + 15 =$ $4x^2 + 32x + 15$
$\mathbf{(2x + 3)(2x + 5)}$	$\mathbf{4x^2 + 10x + 6x + 15 =}$ $\mathbf{4x^2 + 16x + 15}$

The last row of the table yields a product that is the same as the original expression; therefore, $(2x + 3)(2x + 5)$ is the correct factorization of $4x^2 + 16x + 15$. As with the tables in Lesson 12-4, once the result is found, the table need not be completed. The table simply helps the student organize the possible factors and ensures that the necessary factor pairs are examined. This method may be applied to all quadratic trinomials, including those in Lesson 12-4.

Common Error Alert

When the list of possible factors is long, students may tire of performing all the multiplication necessary to find the correct product. They may find the process less tedious if they guess which set of trial factors might work rather than multiply every set of factors.

The tedious nature of the guess-and-check method might encourage some students to look for a way to avoid some of the multiplying. The alternative method discussed next may enable those students to systematically choose which trial factors to try first.

Another method for factoring quadratic trinomials of the form $ax^2 + bx + c$, where $a \neq 1$and $a \neq 0$, combines the guess-and-check method with the method that uses a list of factor pairs and their sums (Lesson 12-4). In Lesson 12-4, factor pairs of the constant term were listed and examined to determine which pair had a sum equal to the coefficient of the middle term. In this lesson, the x^2 term has a coefficient other than one, and that coefficient, also, must be taken into consideration.

Consider the example used earlier, $4x^2 + 16x + 15$. Multiply the coefficient of x^2, 4, by the constant, 15, to get $(4)(15) = 60$. Then use that product to create a table of factor pairs. As in the previous lesson, find the sum of each factor pair.

Factor pair	Sum
(60)(1)	61
(30)(2)	32
(15)(4)	19
(10)(6)	**16**

The correct pair is the one whose sum is the coefficient of the middle term; in this case, we are looking for a factor pair with a product of 60 and a sum of 16. The correct pair is 10 and 6. Therefore, $10x$ and $6x$ should be the result of the multiplication done in the Inner and Outer steps when using the FOIL Method.

Place the factor pairs of $4x^2$ as the first terms of possible binomial factors. This will result in two sets of possible factors:

$$(x + ?)\ (4x + ?) \quad \text{or} \quad (2x + ?)\ (2x + ?).$$

The numbers that will be placed in the second position in the binomials must be factor pairs of the constant term of the trinomial. List the factor pairs of the constant term, 15. Possible factor pairs are (15)(1) and (5)(3). Consider the first binomial pair above. In this pair, one of the factors of 15 will be multiplied by $4x$: however, none of the possible products will result in either $10x$ or $6x$. Therefore, $(x + ?)\ (4x + ?)$ cannot be used. Using the second set of binomial factors, $(2x + ?)\ (2x + ?)$, the student can now insert, as the second term in each binomial, the factor pairs of 15 and check to find the correct one. An element of guess-and-check is still present, but this adaptation of the method may be more efficient for some students.

Common Error Alert

The signs of the coefficients and the constant term must be preserved when finding the product of a and c. When listing the factor pairs, a negative product will result in the table of trial factors being expanded to include all possible combinations of the signs as well. When the sum of the factor pairs is found, the signs must be used, and when those sums are compared with the coefficient of the x term, the sign must be considered also.

1 **Factor: $5m^2 - 4m - 1$.** The only factors of the first term are m and $5m$. Because the last term is negative, its factor pairs must have one positive and one negative value. Here, the factors of the last term are –1 and 1. List the two possible combinations of factors and use the FOIL Method to multiply the binomials. The correct factorization is $(m - 1)(5m + 1)$. It is acceptable to reverse the order of the binomials, provided that no changes are made within the parentheses. Thus, $(5m + 1)(m - 1)$ is also an acceptable result.

Trial Factors	Product
$\mathbf{(m - 1)(5m + 1)}$	$\mathbf{5m^2 + m - 5m - 1 = 5m^2 - 4m - 1}$
$(m + 1)(5m - 1)$	$5m^2 - m + 5m - 1 = 5m^2 + 4m - 1$

Common Error Alert

In this lesson, the tables of factors have not included some factors because those factors would not work—usually because of incorrect signs. Some students may need to see ALL possible factors listed and products calculated, especially at the beginning of the lesson.

Additional Examples

1. Factor, if possible: $3x^2 + 4x + 2$. Use the guess-and-check method.

The factors of the first term are $3x$ and x. The factors of the last term are 1 and 2. Because all of the terms are positive, every term in the binomial pair will be positive. Creating the trial factors and finding their products reveals that no possible combination yields the original expression with a middle term of $4x$. Thus, this trinomial is prime.

Trial Factors	Product
$(x + 1)(3x + 2)$	$3x^2 + 5x + 2$
$(x + 2)(3x + 1)$	$3x^2 + 7x + 2$

2. Factor, if possible: $7x^2 - 15x + 2$. Use the guess-and-check method.

The factors of the first term are x and $7x$. Because the last term is positive, possible factors of that term are 1, 2 and –1, –2. The positive factor pair could not create a middle term that is negative. Therefore, only the negative factors are included in the table.

Trial Factors	Product
$(x - 1)(7x - 2)$	$7x^2 - 9x + 2$
$\mathbf{(x - 2)(7x - 1)}$	$\mathbf{7x^2 - 15x + 2}$

The factor pair whose product is the original expression is $(x - 2)(7x - 1)$, so it is the correct factorization.

Section 2

Expand Their Horizons

In Section 2, students will again factor quadratic trinomials with a leading coefficient other than one. That is, they will factor trinomials of the form $ax^2 + bx + c$, where $a \neq 0$ and $a \neq 1$. In this section, students will use the grouping method that was first demonstrated in Lesson 12-2. The guess-and-check method can be very tedious, especially for trinomials whose first and last terms have many factor pairs. The grouping method can be more efficient.

The grouping method requires a polynomial with at least four terms. So, to begin, the trinomial must be rewritten as a polynomial with four terms. When multiplying binomials using the FOIL Method, a four-term polynomial is the product created before like terms are combined to form the x term. To use the grouping method when factoring, the x term must "uncombined" into two x terms.

Begin by multiplying the coefficient of the x^2 term by the constant term. List factor pairs for this product and find the sum of each pair. This process is similar to the process used in Lesson 12-4. Use the factor pair whose sum is the coefficient of the x term in the trinomial as the coefficients of the two like terms. Then apply the grouping method.

Consider the expression used earlier in this lesson, $4x^2 + 16x + 15$. The product of the coefficient of the x^2 term, 4, and the constant term, 15, is 60. List the factor pairs of 60. Then find the sums of each pair. A sum of 16, the coefficient of the middle term, is the desired outcome. (A table showing this is in Section 1.) The factor pair 10 and 6 result in a product of 60 and a sum of 16. Use those factors as the coefficients of x and rewrite the middle term as $10x + 6x$.

$$4x^2 + 16x + 15 = 4x^2 + 10x + 6x + 15$$

This expression can now be factored by grouping. Group the terms into two binomials so that the terms within each binomial have a common factor.

$$4x^2 + 10x + 6x + 15 = (4x^2 + 10x) + (6x + 15)$$

Factor out the greatest common factor (Lesson 12-1).

$$(4x^2 + 10x) + (6x + 15) = 2x(2x + 5) + 3(2x + 5)$$

Note that there is a common binomial factor, $(2x + 5)$, in each resulting term. Factor out this common factor.

$$2x(2x + 5) + 3(2x + 5) = (2x + 5)\ (2x + 3)$$

Compare the resulting factorization with that obtained earlier when using the guess-and-check method.

After a trinomial is rewritten as a four-term polynomial, it can always be grouped. If the expression were prime, no factor pair with the appropriate product and sum would be found, and the trinomial could not be rewritten for grouping. If a trinomial with leading coefficient other than one is prime, students will be able to determine this immediately from the factor pair list.

Also worthy of note is the fact that the order of the two middle terms is not relevant to the outcome of the factorization. In the previous example, if $4x^2 + 16x + 15$ were rewritten as the expression $4x^2 + 10x + 6x + 15$ with the middle terms reversed, the result would be the same. In this case, $2x$ is the common factor of the first two terms, and 3 is the common factor of the last two terms. Removing these common factors gives $2x(2x + 5) + 3(2x + 5)$ which has a common binomial of $(2x + 5)$ with a final factorization of $(2x + 5)(2x + 3)$; this is the same result.

The sign rules for multiplication and addition apply to the factors listed in the table. Consider the quadratic trinomial $8x^2 - 18x - 5$. The product of the coefficient of the first term, 8, and the constant term, –5, is –40. To get a negative product, one of the factors must be negative and the other positive. There are eight such factor pairs: (1)(–40), (–1)(40), (2)(–20), (–2)(20), (4)(–10), (–4)(10), (5)(–8), (–5)(8). The sum we are looking for is –18. The correct factor pair is 2 and –20 because $(2)(-20) = -40$ and $(2) + (-20) = -18$. Using these as coefficients of the "uncombined" middle term, the expression is rewritten as $8x^2 + 2x - 20x - 5$.

Grouping results in $(8x^2 + 2x) - (20x + 5)$. Recall from Lesson 12-2 that a negative before a set of parentheses changes the signs of the terms within it. The common factor of the first binomial is $2x$, and the common factor of the last binomial is 5. Removing the common factors gives $2x(4x + 1) - 5(4x + 1)$. The common binomial factor can then be factored out, and the resulting factorization is $(4x + 1)(2x - 5)$.

Some students may prefer the grouping method when factoring quadratic trinomials; however, it does involve remembering an additional set of rules. While guess-and-check is tedious, it is more clearly related to the multiplication of binomials and may be easier for students to remember. Students should be made aware that the factored result is the same regardless of which of these methods is used.

2 **Factor: $3x^2 + x - 10$.** The product of the leading coefficient, 3, and the constant term, –10, is –30. List the factor pairs of –30, keeping in mind that each factor pair must contain one positive and one negative number so that the product will be negative. From this list, choose the pair with a sum of one, the coefficient of the middle term x. The correct pair is –5 and 6. Using this factor pair as coefficients of the middle terms, the expression can be rewritten as $3x^2 - 5x + 6x - 10$. When grouped, the expression becomes $(3x^2 - 5x) + (6x - 10)$. When the common factor of each binomial is factored out, the expression becomes $x(3x - 5) + 2(3x - 5)$. The common binomial factor can then be factored out, and the resulting factorization is $(3x - 5)(x + 2)$.

3 **Factor: $8d^2 + 10d - 25$.** The product of the leading coefficient, 8, and the constant term, –25, is –200. List the factor pairs of –200, keeping in mind that each factor pair must contain one positive and one negative number so that the product will be negative. From this list, choose a pair with a sum of 10, the coefficient of the middle term $10d$. That pair is –10 and 20. The expression can be rewritten as $8d^2 - 10d + 20d - 25$, using this factor pair as coefficients of the middle terms. When grouped, the expression becomes $(8d^2 - 10d) + (20d - 25)$. The common factors can be factored out yielding $2d(4d - 5) + 5(4d - 5)$. When the common binomial factor, $(4d - 5)$, is taken out, the factorization is $(4d - 5)(2d + 5)$. This can also be written as $(2d + 5)(4d - 5)$.

Look Beyond

In Lesson 12-6, various methods of factoring from this and previous lessons will be combined as students choose which method is best suited to factoring a given polynomial. Students will also factor polynomials requiring more than one factoring method. All the methods learned in this module will be used in Algebra II, College Algebra, and beyond.

Additional Examples

1. Factor, if possible: $4x^2 + 3x - 18$. Use the grouping method.

Multiply the leading coefficient, 4, by the constant, –18, to get the product, –72. Create a factor list that contains pairs with one positive and one negative factor. To rewrite the expression for grouping, find a factor pair with a sum of three.

Because no such pair exists, this polynomial cannot be factored by the grouping method. In fact, the polynomial is prime.

2. Factor, if possible: $8x^2 - 19x + 6$. Use the grouping method.

Multiply the leading coefficient, 8, by the constant term, 6, to get the product 48. To rewrite the expression for grouping, find a factor pair with a sum of –19. To get a positive product and a negative sum, both factors in the pair must be negative.

The factor pair –3 and –16 provides the correct sum and product. Rewriting the expression as a four-term polynomial, using the factor pair as coefficients of the middle terms, results in the expression $8x^2 - 3x - 16x + 6$. If we group the terms as $(8x^2 - 3x) - (16x - 6)$, the signs of the terms within the second binomial change because of the negative sign placed in front of it. Factoring out the common factors yields $x(8x - 3) - 2(8x - 3)$, which becomes $(8x - 3)(x - 2)$ when the common binomial factor is factored out.

Manipulatives

The manipulatives used in this lesson are algebra tiles. The procedure for using these tiles to represent a given trinomial and determine its binomial factors is exactly the same as that for Lesson 12-4. Refer to that lesson for a complete explanation of using algebra tiles. Note: Just as the 1's tiles were always placed in a rectangular or square shape, the x^2 tiles must always be placed in a rectangular or square shape. Although Problem 4, $4x^2 - 8x - 12$, can be factored using algebra tiles as $(4x + 4)(x - 3)$, it is not factored completely. A GCF of 4 should be removed first; as will be explained in Lesson 12-6.