12.4 teacher notes

Objective

- Factor trinomials of the form $x^2 + bx + c$.

Vocabulary

Factoring (Lesson 12-1)
Quadratic Term
Coefficient (Lesson 2-1)
FOIL Method (Lesson 11-4)
Like terms (Lesson 2-4)

Prerequisites

Multiplying binomials by the FOIL Method

Applying the laws of exponents

Get Started

- Divide the students in the class into groups or pairs. Assign each group one of the expressions below. Tell the students to use the FOIL Method to rewrite their expression. A representative of each group should share the group's answer with the class.

Expression	Expression Rewritten
$(x + 3)(x + 8)$	$x^2 + 11x + 24$
$(y - 1)(y - 6)$	$y^2 - 7y + 6$
$(r - 2)(r + 5)$	$r^2 + 3r - 10$
$(b - 5)(b + 4)$	$b^2 - b - 20$
$(a + 3)^2$	$a^2 + 6a + 9$

- Explain to students that in each case, they rewrote a product of binomials as a trinomial. For example, the product $(x + 3)(x + 8)$ was rewritten as the trinomial $x^2 + 11x + 24$.
- In this lesson, students will learn to rewrite trinomials as the product of two binomials. For example, they will rewrite a trinomial such as $x^2 + 11x + 24$ as the product $(x + 3)(x + 8)$. This is another method of factoring.

- Point out that whether they are multiplying $(x + 3)(x + 8)$ to get $x^2 + 11x + 24$ or factoring $x^2 + 11x + 24$ to get $(x + 3)(x + 8)$, they can use the FOIL Method in both factoring and multiplying. The product of the first terms in each binomial will result in the first term of the product, and the product of the last terms in each binomial will result in the last term of the product. The product of the outer term and inner term of the binomial will be combined if they are like terms to form the middle term of the product. When students are going from the product to the two binomial factors, encourage them to ask, "Which two terms in the binomial when multiplied resulted in the first term of the product?" Proceed to ask about the formulation of the last term and the middle term.

Section 1

Expand Their Horizons

In Section 1, students will factor quadratic trinomials with a leading coefficient of one and positive, real coefficients and constants. Quadratic trinomials are three-term expressions that have a degree of two and can be written as $ax^2 + bx + c$, where a, b, and c are real numbers. A leading coefficient of one means that $a = 1$. The trinomials studied in this section are of the form $1x^2 + bx + c$ or of the simplified $x^2 + bx + c$, where $b > 0$ and $c > 0$.

In this lesson students will use the FOIL Method, applied in reverse, to factor quadratic trinomials. In years past, the standard method for factoring these expressions has been the *trial-and-error method*. This method, also called the *guess-and-check method*, requires students to repeatedly guess at binomial pairs and find their product until the correct factorization is found. The guess and check method is mentioned in Lesson 12-5. However, as explained in both this lesson and the next, much more methodical and reliable methods are now being used.

Recall that two binomials can be multiplied together using the FOIL Method: First, Outer, Inner, Last. Also, recall that the outer and inner terms are often like terms that can be combined.

For example, consider $(x + 3)(x + 5)$. Using the FOIL Method, the product is $x^2 + 5x + 3x + 15$, which simplifies to $x^2 + 8x + 15$. Note that the three and five in the binomials can be multiplied to yield 15 (the last term in the trinomial product) and added to yield eight (the coefficient of the middle term in the trinomial product). This concept will be applied to reverse the process and factor. Given the trinomial $x^2 + 8x + 15$, it must be determined what pair of numbers can be multiplied to yield 15 and be added to yield eight. The answer is three and five, so the Last terms of the binomial pair will be three and five. Determining the correct variables is easier. To get x^2, each binomial must have an x as the First term: Thus, the factored result is $(x + 3)(x + 5)$.

Consider the trinomial $x^2 + 22x + 40$. Begin by determining what pair of numbers can be multiplied to get 40 and added to get 22. A table will be helpful. In the table, list the factors of the constant (last) term and then find their sums (corresponding to the coefficient of the middle term). In this example, list the factors of 40 and find their sums to determine which pair has a sum of 22.

Factors of 40	Sum
$1 \cdot 40$	41
$\mathbf{2 \cdot 20}$	**22**
$4 \cdot 10$	14
$5 \cdot 8$	13

The correct choice is two and 20. Thus, this trinomial factors as $(x + 2)(x + 20)$. Note that because multiplication is commutative, the order of the binomials in the factored result does not matter. That is, $(x + 20)(x + 2)$ is also an acceptable factorization of $x^2 + 22x + 40$.

Common Error Alert

Students may mistakenly label a trinomial as prime because they did not find the pair with the correct sum although it existed. To determine that all possible factor pairs have been listed in the table, encourage students to be methodical. Start with one times the given number, and then progress to two times some value, and then three times some value, etc. Students with weak arithmetic skills may use a calculator to divide the given number by successive numbers. A decimal result signifies that the number did not divide evenly and must be discarded. Consider the previous example. Because $48 \div 1 = 48$, the factor pair $1 \cdot 48$ is listed. Because $48 \div 2 = 24$, $2 \cdot 24$ is listed. However, when the student gets to division by five, $48 \div 5 = 9.6$, so five is not a factor of 48; five is not placed in the table, and the student progresses to the next number, which is six. When the student reaches a number that is already listed in the table, the table is complete. Again consider the previous example. After dividing by six to find that $6 \cdot 8$ is a factor pair of 48, the student will progress to seven (which is not a factor) and then to eight. Because eight is already listed as the "partner" of six, all possible factor pairs have been found, and the list is complete. If the pair with the desired sum is not present, the student can feel confident that the polynomial cannot be factored by this method.

Encourage students to check their factorizations. To check, use the FOIL Method to multiply the binomials. If the product is the same as the original trinomial expression, then the factorization is correct. Checking the previous example, $(x + 2)(x + 20) = x^2 + 20x + 2x + 40 = x^2 + 22x + 40$.

This method may be applied to other trinomials, such as $x^4 + 22x^2 + 40$, which factors as $(x^2 + 2)(x^2 + 20)$. However, all expressions that require factoring should be checked for common factors first. In this lesson, no such common factors will be found; multiple factoring methods will be examined in Lesson 12-6.

Consider the expression $x^2 + 12x + 36$. List the factors of 36 and find which pair of factors has a sum of 12.

Factors of 36	Sum
$1 \cdot 36$	37
$2 \cdot 18$	20
$3 \cdot 12$	15
$4 \cdot 9$	13
$6 \cdot 6$	**12**

The correct choice of factors is six and six because $6 \cdot 6 = 36$ and $6 + 6 = 12$. The factored representation is $(x + 6)(x + 6)$, which can be written as $(x + 6)^2$ because multiplying a term by itself is the same as squaring it.

Finally, consider the example $x^2 + 20x + 48$. Create a table of factors of 48 and their sums.

Factors of 48	Sum
$1 \cdot 48$	49
$2 \cdot 24$	26
$3 \cdot 16$	19
$4 \cdot 12$	16
$6 \cdot 8$	14

There is no factor pair of 48 whose sum is 20; therefore, this trinomial cannot be factored according to the method in this lesson. This trinomial is *prime*. Note that for any polynomial to be classified as prime, it must be determined that *none* of the factoring methods can be applied to it. Such is the case with this example, but it may be preferable for students to declare a problem such as this as "not factorable by this method" until they have completed all sections of this module.

1 **Factor $y^2 + 10y + 9$.** List the pairs of factors of nine and look for a pair with a sum of 10. That pair is one and nine. To get y^2, the variable y is placed in each binomial. The result is $(y + 1)(y + 9)$.

Factors of 9	Sum
$1 \cdot 9$	**10**
$3 \cdot 3$	6

2 **Factor: $t^2 + 2t + 1$.** The only factor pair for one is $1 \cdot 1$, and $1 + 1 = 2$. To get t^2, the variable t is placed in each binomial. The result is $(t + 1)(t + 1)$ or $(t + 1)^2$.

Common Error Alert

Students may feel that the table of factors needs to be finished with all possible factor pairs listed. The table is simply an organized way to find the factor pair with the correct sum. Should students find the correct pair before the table is "finished," they should proceed to writing the binomial factors and disregard the rest of the table. Although some students can find the pairs mentally, encourage them to double check their pair by completing the factor table and acknowledging that there are no other pairs that work.

Additional Examples

1. Factor, if possible: $x^2 + 13x + 30$.

List the factors of 30 to find a pair with a sum of 13. That pair is three and 10.

Factors of 30	Sum
$1 \cdot 30$	31
$2 \cdot 15$	17
$3 \cdot 10$	**13**
$5 \cdot 6$	11

The factorization is $(x + 3)(x + 10)$. Checking, FOIL multiplication yields $x^2 + 10x + 3x + 30$, which simplifies to $x^2 + 13x + 30$. Because this is the original problem, the factorization is correct.

2. Factor, if possible: $r^2 + 12r + 18$.

List the factors of 18 to find a pair with a sum of 12.

Factors of 18	Sum
$1 \cdot 18$	19
$2 \cdot 9$	11
$3 \cdot 6$	9

There is no such pair. Thus, the trinomial is prime.

Section 2

Expand Their Horizons

In Section 2, students will expand the concept of applying the FOIL Method in reverse to include those trinomials with negative second and/or third terms. Still, only those quadratic trinomials with a leading coefficient of one are examined. Thus, the trinomials in this section are of the form $x^2 + bx + c$, where $b < 0$ and/or $c < 0$.

First, consider $x^2 + 8x + 12$. According to Section 1, a list of the factors of 12 can be used to find a factor pair whose sum is eight. That pair is two and six, resulting in the factorization $(x + 2)(x + 6)$. By contrast, consider $x^2 - 8x + 12$. A list of the factors of 12 is still useful, but because a negative sum is being sought, only pairs in which both factors are negative should be considered. The two negative factors will yield a positive product, the 12 and a negative sum.

Factors of 12	Sum
(–1)(–12)	–13
(–2)(–6)	**–8**
(–3)(–4)	–7

The pair –2 and –6 yields a sum of –8. Therefore, the factorization is $(x - 2)(x - 6)$. Again, the order of the binomials is not significant, and the result could also be written as $(x - 6)(x - 2)$. Students should use the FOIL Method to check the factorization: $(x - 6)(x - 2) = x^2 - 2x - 6x + 12 = x^2 - 8x + 12$.

Consider $x^2 + 7x - 18$ and $x^2 - 7x - 18$. In both, the last term is negative. Recall the last term is the one from which factor pairs are created. To multiply two numbers and get a negative product, one number must be negative and the other positive. Thus, each pair of factors will be listed twice in the table in order to indicate all possible combinations of the signs. Further, in the first expression, a sum of 7 is to be found, but in the second expression a sum of –7 is to be found.

Factors of 18	Sum
(–1)(18)	17
(1)(–18)	–17
(–2)(9)	**7**
(2)(–9)	**–7**
(–3)(6)	3
(3)(–6)	–3

The sum of +7 is found with the pair –2 and 9; thus, $x^2 + 7x - 18$ factors as $(x - 2)(x + 9)$. The sum of –7 is found with the pair +2 and –9; thus, $x^2 - 7x - 18$ factors as $(x + 2)(x - 9)$.

It should be noted that while the order of the binomials may be different, the signs within each binomial expression may not change. The factorization for $x^2 - 7x - 18$ can be written $(x + 2)(x - 9)$ or $(x - 9)(x + 2)$.

Common Error Alert

Sign mistakes are the most common student errors in factoring trinomials. In the expression $x^2 + bx + c$, $c > 0$ means that both numbers in the binomial have the same sign, and that sign is determined by b:

- $x^2 + bx + c$, $c > 0$ and $b > 0$, then both numbers in the binomials are positive.
- $x^2 + bx + c$, $c > 0$ and $b < 0$, then both numbers in the binomials are negative.

But in the expression $x^2 + bx + c$, $c < 0$ means that one number in the binomial is positive and the other is negative, and the sign of the number with the larger absolute value is determined by b:

- $x^2 + bx + c$, $c < 0$ and $b > 0$, then the number with the greater absolute value in the pair is positive.
- $x^2 + bx + c$, $c < 0$ and $b < 0$, then the number with the greater absolute value in the pair is negative.

Recall that adding numbers with like signs is done by adding the numbers and keeping their common sign. Adding numbers with unlike signs is the same as subtracting. It may be preferable to divide the problems in this lesson into two groups: Those whose last term is positive (requiring addition to determine the coefficient of the middle term) and those whose last term is negative (requiring subtraction to determine the coefficient of the middle term). See the four examples below for a demonstration of each case.

$x^2 + 5x + 6$

Factors of 6	Sum
(1)(6)	7
(2)(3)	**5**
(–1)(–6)	–7
(–2)(–3)	–5

like signs
both positive
$(x + 2)(x + 3)$

$x^2 - 5x + 6$

Factors of 6	Sum
(1)(6)	7
(2)(3)	5
(–1)(–6)	–7
(–2)(–3)	**–5**

like signs
both negative
$(x - 2)(x - 3)$

$x^2 + 5x - 6$

Factors of –6	Sum
(–1)(6)	**5**
(–2)(3)	1
(1)(–6)	–5
(2)(–3)	–1

unlike signs
larger positive
$(x - 1)(x + 6)$

$x^2 - 5x - 6$

Factors of –6	Sum
(–1)(6)	5
(–2)(3)	1
(1)(–6)	**–5**
(2)(–3)	–1

unlike signs
larger negative
$(x + 1)(x - 6)$

An alternative method, using grouping, can be used to factor all the trinomials in this lesson. This method will be explained in the second section of Lesson 12-5.

3 **Factor: $r^2 - 9r + 20$.** Because the last term (which indicates the product) is positive and the middle term (which indicates the sum) is negative, the factors must both be negative.

Factors of 20	Sum
(–1)(–20)	–21
(–2)(–10)	–12
(–4)(–5)	**–9**

The pair –4 and –5 has a sum of –9. Thus, the factorization is $(r - 4)(r - 5)$. To check, multiply the binomial factors. The result is $r^2 - 5r - 4r + 20$, which simplifies to $r^2 - 9r + 20$, demonstrating that the factorization is correct.

4 **Factor: $k^2 + 8k - 20$.** Because the last term is negative, the product is negative and must, therefore, be formed by one positive and one negative factor.

Factors of –20	Sum
(–1)(20)	19
(1)(–20)	–19
(–2)(10)	**8**
(2)(–10)	–8
(–4)(5)	1
(4)(–5)	–1

The pair whose sum is +8 is –2 and +10. Thus the factorization is $(k - 2)(k + 10)$. To check, multiply the binomial factors. The result is $k^2 + 10k - 2k - 20$, which simplifies to $k^2 + 8k - 20$, demonstrating that the factorization is correct.

Look Beyond

What happens with an expression such as $5x^2 + 13x - 6$, which has a leading coefficient other than one? These expressions will be factored in Lesson 12-5, requiring the same thought process but additional steps to complete. Other expressions such as $x^4 + 22x^2 + 40$, $x^{\frac{2}{3}} + 8x^{\frac{1}{3}} + 15$, or $x^{-2} + 7x^{-1} - 18$ can be factored by this method but are usually encountered in Algebra II or college algebra.

Additional Examples

1. Factor, if possible: $y^2 + 13y - 30$.

List the factors of –30 to find a pair with a sum of +13. Note that one factor of the pair must be positive and the other negative to yield a negative product.

Factors of –30	Sum
(–1)(30)	29
(1)(–30)	–29
(–2)(15)	**13**
(2)(–15)	–13
(–3)(10)	7
(3)(–10)	–7
(–5)(6)	1
(5)(–6)	–1

The correct factor pair is –2 and +15. Thus, the factorization is $(y - 2)(y + 15)$. FOIL multiplication yields $y^2 + 15y - 2y - 30$, which simplifies to $y^2 + 13y - 30$. Because this is the original problem, the factored result checks.

2. Factor, if possible: $x^2 - 6x + 7$.

List the factors of 7 to find a pair with a sum of –6. Both factors of each pair must be negative to get a negative sum and a positive product. The only factors that are negative and yield a product of +7 are –1 and –7, which have a sum of –8. It is not possible to factor this trinomial by the method in this lesson; there is no such pair. (Incorrect application of signs makes +1 and –7 seem like the correct factors, yet these have a product of –7 not +7). This trinomial is prime.

Manipulatives

Refer to Lesson 11-4 for multiplication of two binomials using algebra tiles. That procedure will be used in reverse for the factoring of a quadratic trinomial into two binomials. In reviewing Lesson 11-4, note the position of the x^2 tiles in the upper left corner and the position of the one tiles in the lower right corner. The placement will be the same for factoring. Again, the small square tiles represent one's (constants), with the yellow side being positive and the red side being negative. The rectangular tiles represent x's, with the green side being positive and the red side being negative. The large square tiles represent x^2's, with the blue side being positive and the red side being negative.

Consider the trinomial $x^2 + 7x + 6$. A large blue square tile is placed in the upper left corner to represent x^2, and six small, square yellow tiles are placed in the form of a rectangle in the lower right corner to represent +6. In this example, there are two choices for the placement of these yellow tiles: in a 2 by 3 rectangle or a 1 by 6 rectangle. Figure 1 shows the placement as a 2 by 3 rectangle. As more than one choice is available and each must be attempted to find the correct solution, this is a representation of the guess and check method.

Use seven rectangular green tiles to represent $+7x$. These must be placed below and to the right of the x^2 tile so that the entire set of tiles creates a rectangle. All tiles must be used, and no empty spaces may exist. Because placing the green tiles does not use them all (see Figure 2), another attempt must be made. This time, the yellow tiles will be placed in a 1 by 6 rectangle as represented by Figure 3. Then, the seven green tiles are placed in the vacant spots as shown in Figure 4. All seven green tiles are used, and the result is a large rectangle containing all the tiles.

Figure 1

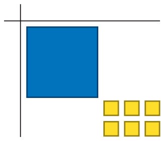


Figure 2

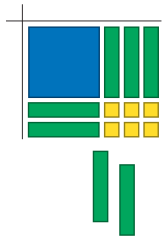


Figure 3

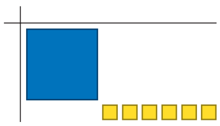


Figure 4

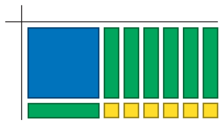
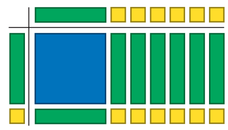


Figure 5



Now, the resulting factorization can be determined and read. Above and to the left of the gridlines, place rectangular and small square tiles so that the products are the tiles in the formation just created. A large, square blue tile is created by the multiplication of two green rectangular tiles (all representing positive values); these are placed above and to the left of the gridlines. Small, square yellow tiles are placed in the other vacant spots above and to the left of the gridlines (again, representing positive values). See Figure 5. To the left of the gridline is one x tile and one 1's tile; this is $(x + 1)$. Above the gridline is one x tile and six 1's tiles; this is $(x + 6)$. The factored form of the trinomial $x^2 + 7x + 6$ is $(x + 1)(x + 6)$. This method reverses the process learned in Lesson 11-4: the mulitplication of binomials using algebra tiles. In the DVD lesson, the gridlines are omitted, and students simply find the length and width of the rectangle by inspection.

When the coefficient of x and/or the constant term is negative, the addition of zero pairs may be necessary. Consider the trinomial $x^2 + 3x - 10$. This expression is represented by one large, square blue tile (x^2), three green rectangular tiles ($+3x$), and ten small, square red tiles (–10). The x^2 and 1's tiles are laid first, according to Figure 6. The 1's tiles could be placed in a 1 by 10 rectangle or a 2 by 5 rectangle; only the correct choice of 2 by 5 is shown here. Then, the three green rectangular tiles are placed. Figure 7 illustrates this placement. The tiles do not make a full rectangle, and no alternate placement offers such. When the number of tiles is less than the number of vacant spaces, place the tiles in the larger of the two "holes." In this example, there is a hole in the upper right and lower left; the larger hole is upper right, so the three green rectangular tiles are placed there.

Then, zero pairs are added. There are four empty rectangular slots, so two red (for $-2x$) and two green (for $+2x$) are added. Since $(-2x) + (+2x) = 0$, the addition of these tiles does not change the value of the expression. Note that the $+2x$ green tiles are placed with the other green tiles and the $-2x$ red tiles are placed along the left side. See Figure 8 for an illustration.

Figure 6

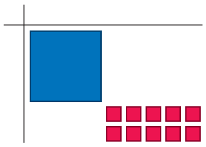


Figure 7

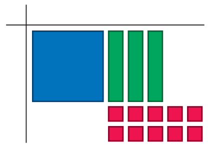


Figure 8

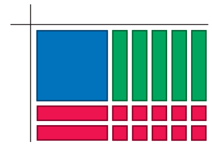


Figure 9

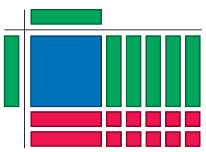


Figure 10

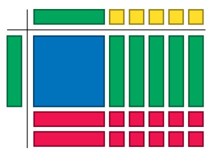
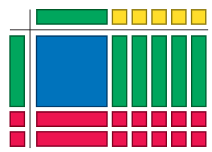


Figure 11



The large, square blue tile is formed by the multiplication of two rectangular green tiles; that is, x^2 is the product of x and x. These two rectangular green tiles are placed above and to the left of the gridline. See Figure 9. The newly placed, rectangular green tile on the left of the gridline must be multiplied by $+1$ values above the gridline to yield $+x$, green tiles within. These values are denoted by small, square yellow tiles as shown in Figure 10. To get the red tiles,which are negative, both rectangle and small square, the positive tiles above the gridline must be multiplied by -1 values to the left of the gridline. These are denoted with small, square red tiles as shown in Figure 11.

Reading the units to the left of the gridline, the rectangular green tile represents x, and the two small, square red ones represent -2 for a binomial of $(x - 2)$. Across the top of the gridline, the rectangular green tile represents x, and the five small, square yellow ones represent $+5$ for a binomial of $(x + 5)$. The factorization is $(x - 2)(x + 5)$. This method completely reverses the manipulative lesson 11-4 of the mulitplication of binomials using algebra tiles. In the DVD lesson, the gridlines are omitted, and students simply find the length and width of the rectangle by inspection.