12.2 teacher notes

Objectives

- Factor out a common binomial factor.
- Factor by *grouping*.

Vocabulary

Factoring (Lesson 12-1)
Grouping
Common factor (Lesson 12-1)
Greatest common binomial factor (Lesson 12-1)
Distributive Property (Lesson 2-3)

Prerequisites

Factoring by removing the greatest common factor

Applying the Distributive Property

Multiplying two binomials

Using the *laws of exponents*

Finding the area of a rectangle

Get Started

- Assemble the following: five blue blocks, two yellow blocks, four red blocks, five green blocks, and two identical boxes for holding blocks. (Items other than blocks may be used.)
- Make two piles of blocks on a table. In the first pile, place three blue blocks, one red block, and one green block. In the second pile, place two blue blocks, two yellow blocks, and one green block.
- Ask the students, “What blocks are common to both piles?” Students will reply that two blue blocks and one green block are common. As each is mentioned, remove that block from each pile. That is, when a student states that both piles have a blue block, remove a blue block from each pile, and so forth. Ask students to think of the blocks as monomial factors so that removing common blocks from the piles is like “removing” a common monomial factor from a polynomial.

- Make two new piles on a table. In the first pile, place one blue block, one green block, and a box that contains one red block and one green block. In the second pile, place two red blocks, one yellow block, and a box that contains one red block and one green block.
- Ask the students, "What are common to both piles?" Students should reply that the boxes of identical blocks are common. Ask students to think of the identical boxes of blocks as a common binomial factor. Tell students that in this lesson, common binomial factors will be factored out of expressions.

Section 1

Expand Their Horizons

In Section 1, students will factor expressions by factoring out the *greatest common binomial factor*. In Lesson 12-1, students factored expressions by factoring out the greatest common monomial factor. Help students realize that one skill which is required in factoring is to be able to identify common factors, whether those common factors are monomials or binomials (or even polynomials of more than two terms).

Recall that multiplying two binomials is based on the Distributive Property. For example, the first step in expanding the expression $(x + 2y)(x - 3)$ might be $x(x - 3) + 2y(x - 3)$. Factoring out the greatest common binomial factor is the reverse of this first step. To factor the expression $x(x - 3) + 2y(x - 3)$, the common binomial factor $(x - 3)$ is identified. This common binomial factor is placed in front of parentheses containing the remaining factors, so the expression in factored form is $(x - 3)(x + 2y)$. The expression in factored form could also be $(x + 2y)(x - 3)$ by the Commutative Property.

Consider the expression $2x(3x + 2) + (3x + 2)$. The common factor is the binomial $(3x + 2)$. Rewriting the original expression as $2x(3x + 2) + 1(3x + 2)$ helps the student to see that the factored result is $(3x + 2)(2x + 1)$. If students have trouble with this technique, use the technique from the second section of Lesson 12-1. Removing the common binomial factor from $2x(3x + 2) + (3x + 2)$, forming an expression with that common binomial factor as a denominator, and then, simplifying, we have $2x(3x + 2) + (3x + 2) =$

$(3x + 2)\left[\frac{2x(3x + 2) + (3x + 2)}{(3x + 2)}\right] =$

$(3x + 2)\left[\frac{2x(3x + 2)}{(3x + 2)} + \frac{(3x + 2)}{(3x + 2)}\right] =$

$(3x + 2)(2x + 1)$.

Consider the expression $x(x + 4) - 2(x - 4)$. There is no common factor. The two binomials that appear, $(x + 4)$ and $(x - 4)$, both contain an x and a four, but the binomials are different because of the signs. This expression cannot be factored.

1 **Factor: $x(y + 4) - 3(y + 4)$.** The common binomial factor is $(y + 4)$. Factoring this out of the first term leaves x, and out of the second term leaves -3. The factored expression is $(y + 4)(x - 3)$.

Additional Examples

1. Factor, $a(x + y) - b(y + x) + (x + y)$.

Note that the binomial factor $(y + x)$ in the second term can be rewritten as $(x + y)$, so the factor common to all three terms is $(x + y)$. Note, too, that the third term has a one as its understood coefficient. Thus, the expression can be written as $a(x + y) - b(x + y) + 1(x + y)$. Although this expression has more terms than do the other expressions in this lesson, the technique used to factor it is the same: the common binomial factor is placed in front of parentheses containing the remaining factors. The result is $(x + y)(a - b + 1)$.

2. Factor $(x + 3)^2 + 5(x + 3)$.

The factor common to both terms is the binomial $(x + 3)$. Recall that the lower power of a common variable, or in this case variable expression, is the most that can be factored out. Therefore, $(x + 3)^1$ is factored from each term. Factoring $(x + 3)^1$ out of the first term leaves $(x + 3)$; factoring $(x + 3)^1$ out of the second term leaves five. The result is $(x + 3)[(x + 3)+ 5]$. Combining like terms within the brackets, the simplified result is $(x + 3)(x + 8)$.

Section 2

Expand Their Horizons

In Section 2, students will factor by *grouping*. Grouping can often be applied to polynomials with more than three terms. First, groups are formed so that each group has a common factor. A different greatest common factor is factored out of each group. Then, the greatest common binomial factor is factored from each term. In this lesson, the polynomials have four terms; two groups are formed; and then, a common monomial is factored from each group. However, factoring by grouping can also be used for some polynomials with more than four terms; in which case, the common factor may itself have more than two terms.

Consider the example $ab + 2a + bc + 2c$. There is no factor common to all four terms, so there is no GCF other than one. (Students will learn in later lessons that looking for a GCF is the first step in factoring.) Because the expression has more than three terms, try factoring by grouping. First, form groups: $(ab + 2a) + (bc + 2c)$. Second, factor out each group's GCF. For the first group, the GCF is a, and for the second group, the GCF is c. The expression becomes $a(b + 2) + c(b + 2)$.

The third step in factoring by grouping is to factor out the common binomial factor, which in this case is $(b + 2)$. The factored expression is $(b + 2)(a + c)$.

It is worth noting, that in many cases, there is more than one way to group the terms in a polynomial. For $ab + 2a + bc + 2c$, it makes sense to group the terms as shown in the paragraph above. However, if the order of terms is changed, the steps could be as follows: $ab + 2a + bc + 2c = ab + bc + 2a + 2c = (ab + bc) + (2a + 2c) = b(a + c) + 2(a + c) = (a + c)(b + 2)$. This result is equivalent to the previous result, $(b + 2)(a + c)$, by the Commutative Property.

Factoring is checked by multiplying. For a factoring result that has binomial factors, those binomial factors can be multiplied by either using the FOIL Method or by applying the Distributive Property twice as in Lesson 11-4. If the factoring was done correctly, the result should be the original polynomial.

To check the previous example, multiply $(b + 2)(a + c)$. To use the Distributive Property, the first step is to rewrite the expression as $b(a + c) + 2(a + c)$. Then, apply the Distributive Property again. The product becomes $ab + bc + 2a + 2c$. Reorder the terms to get $ab + 2a + bc + 2c$, the original polynomial. The factored result checks.

For some polynomials, the terms will need to be rearranged in order to use the grouping method. The expression $x^2 + 12y + 3xy + 4x$ is such a polynomial. If grouped while in the original order, the expression becomes $(x^2 + 12y) + (3xy + 4x)$. The first group does not have a GCF other than one, and the second group has a GCF of x, so the expression could be written $(x^2 + 12y) + x(3y + 4)$, but this has no common binomial factor. If the terms are rearranged as $x^2 + 3xy + 12y + 4x$, however, they can be grouped as $(x^2 + 3xy) + (12y + 4x)$. Then, factoring out each group's GCF results in $x(x + 3y) + 4(3y + x)$. Note that the binomials, $(x + 3y)$ and $(3y + x)$, are equivalent, and the expression can be written as $x(x + 3y) + 4(x + 3y)$. Factoring out the common binomial, the expression becomes $(x + 3y)(x + 4)$. Often, a polynomial can be rearranged in more than one way to produce common binomial factors. A different rearrangement of this polynomial, along with the remaining steps, could have been: $x^2 + 4x + 3xy + 12y = (x^2 + 4x) + (3xy + 12y) = x(x + 4) + 3y(x + 4) = (x + 4)(x + 3y)$.

Common Error Alert

The second step in factoring by grouping, in which each group's GCF is removed, is *not* the factored result. Students may mistakenly give this intermediate step as the factored result for a polynomial in which there was not a common binomial. In fact, if the polynomial cannot be rearranged and grouped so that a common binomial factor can be removed, it cannot be factored.

Consider the polynomial $x^2 + 7x + 5x + 10$. Grouping this polynomial in the given order yields $(x^2 + 7x) + (5x + 10)$. When each group's GCF is factored out, the expression becomes $x(x + 7) + 5(x + 2)$. Because the binomial factors of each term are different, no more factoring can be done. No rearrangement of the terms yields common binomial factors; thus, $x^2 + 7x + 5x + 10$ cannot be factored.

Consider the polynomial $5x^2 - 10x - 6x + 12$. Because the third term is negative and this is the point at which the polynomial is separated into two groups, the signs must be watched carefully. When the terms are grouped, the polynomial becomes $(5x^2 - 10x) - (6x - 12)$. Continuing, $(5x^2 - 10x) - (6x - 12) = 5x(x - 2) - 6(x - 2) = (x - 2)(5x - 6)$.

Consider the polynomial $8ab + 4a - 2b - 1$. To begin, group terms. Because the third term is negative, place a subtraction sign between the groups and make appropriate sign changes: $(8ab + 4a) - (2b + 1)$. Factoring out the GCF in the first group yields $4a(2b + 1) - (2b + 1)$ or $4a(2b + 1) - 1(2b + 1)$. Factoring out the common binomial $(2b + 1)$, the factored expression is $(2b + 1)(4a - 1)$.

2 **Factor: $ar + as + br + bs$.** The first two terms have a common factor a, and the last two terms have a common factor b. Therefore, the expression can be grouped in its current order, $(ar + as) + (br + bs)$, and the GCF's can be factored out to yield $a(r + s) + b(r + s)$. The binomial $(r + s)$ is common to both terms, and factoring out this binomial yields $(r + s)(a + b)$.

3 **Factor: $10 - xy - 2y + 5x$.** Neither the first two terms nor the second two terms have a common factor, so this polynomial must be rearranged if factoring by grouping is to be attempted. If the polynomial is arranged as $5x - xy + 10 - 2y$, the first two terms have a common factor of x ,and the last two terms have a common factor of two. After grouping, the expression is $(5x - xy) + (10 - 2y)$. After factoring, the expression is $x(5 - y) + 2(5 - y)$ and then $(5 - y)(x + 2)$. Another possible sequence of steps could be $5x + 10 - xy - 2y = (5x + 10) - (xy + 2y) = 5(x + 2) - y(x + 2) = (x + 2)(5 - y)$.

Additional Examples

1. Factor, if possible: $5x^2 - 10x - x - 2$.

Grouping, the expression becomes $(5x^2 - 10x) - (x + 2)$. Note that because the third term in the original expression is negative, a subtraction sign is placed outside the second group, and the signs of both terms within that group are changed. The terms of the first group have a common factor of $5x$, and the terms of the second group have no common factor other than one, so the one is written in front of that group. Factoring out the greatest common factors of the groups yields $5x(x - 2) - 1(x + 2)$. The last step in factoring is to factor out the common binomial factor, but these terms have no common binomial factor. In fact, no rearrangement provides a common binomial factor. Therefore, the polynomial $5x^2 - 10x - x + 2$ cannot be factored.

2. Factor, if possible: $x^2 + xy - 2x - 2y + xy + y^2$. HINT: Form three groups.

If we group the polynomial as pairs of terms, the expression becomes $(x^2 + xy) - (2x + 2y) + (xy + y^2)$. The common factor of the first group is x, of the second group is two, and of the third group is y. Factoring out the common factor from each group yields $x(x + y) - 2(x + y) + y(x + y)$. All three groups have a common binomial factor, $(x + y)$. If this common factor is taken out and the remaining factors are placed within parentheses, the factored expression becomes $(x + y)(x - 2 + y)$.

Look Beyond

Lesson 12-1 addressed factoring by removing the greatest common monomial factor; that technique can be applied to polynomials that have two or more terms. This lesson addressed factoring by grouping, which is appropriate for some polynomials that have four terms or more. The factoring techniques in the remaining sections of Module 12 apply mostly to polynomials of two or three terms.