12.1 teacher notes

Objectives

- Identify the GCF of a polynomial.
- Factor a polynomial by removing the GCF.

Prerequisites

Applying the Distributive Property

Using the *laws of exponents*

Writing a monomial as a product of prime factors

Vocabulary

Factoring
Distributive Property (Lesson 2-3)
Monomial (Lesson 2-1)
Binomial (Lesson 2-1)
Variable (Lesson 2-1)
Polynomial (Lesson 2-1)
Common factor
Greatest common monomial factor (GCF)

Get Started

- Divide the students in the class into groups or pairs. Assign each group one of the expressions below. Tell the students to use the Distributive Property to rewrite their expression. A representative of each group should share the group's answer with the class.

Expression	Expression Rewritten
$2(x - 5)$	$2x - 10$
$a(x + y)$	$ax + ay$
$r(r - 1)$	$r^2 - r$
$3(b^2 + 2b - 5)$	$3b^2 + 6b - 15$
$3ab^2(2a - 3b + ab)$	$6a^2b^2 - 9ab^3 + 3a^2b^3$

- Explain to students that in each case, they rewrote a product as a polynomial. For example, the product $2(x - 5)$ was rewritten as the polynomial $2x - 10$.

- In this lesson, students will learn to rewrite polynomials as products. For example, they will rewrite a polynomial such as $2x - 10$ as the product $2(x - 5)$. This is called factoring a polynomial by removing the *greatest common factor*. The greatest common factor of the polynomial $2x - 10$ is two, which is "removed" and written outside of a set of parentheses.
- Point out that whether $2(x - 5)$ is rewritten as $2x - 10$ or that $2x - 10$ is rewritten as $2(x - 5)$, it is the Distributive Property that is being applied. The *distributive property* states that $a(b + c) = ab + ac$, so it allows $a(b + c)$ to be rewritten as $ab + ac$ or $ab + ac$ to be rewritten as $a(b + c)$.
- The *greatest common monomial factor* in a polynomial is sometimes simply called the *greatest common factor*, because a common factor of the terms in a polynomial is always a monomial. A monomial is either a numeral, a variable, or a product of a numeral and one or more variables. So, the phrases "greatest common factor" and "greatest common monomial factor" are used interchangeably. The abbreviation GCF (for greatest common factor) is also used.

Factoring a polynomial is rewriting the polynomial as a product of simpler expressions. Factoring a polynomial by removing the greatest common factor is only one of several methods of factoring. Other methods will be studied in subsequent lessons. For polynomials that require more than one method of factoring, removing the greatest common factor is the first method to be used.

Section 1

Expand Their Horizons

In Section 1, students will be shown how to identify the greatest common factor of a polynomial. The first step is to write each term of the polynomial in prime factored form. Then, to get the greatest common factor, use the greatest numerical factor that appears in every term and the highest power of any variable that appears in every term. Two examples are illustrated below:

- $30x^2 + 12x - 18$ would be written $\underline{2 \cdot 3} \cdot 5 \cdot x \cdot x + 2 \cdot \underline{2 \cdot 3} \cdot x - \underline{2 \cdot 3} \cdot 3$. The greatest numerical factor that appears in every term is $\underline{2 \cdot 3}$ or 6. There is no variable that appears in every term. So, the GCF is six.
- $27x^2 + 20x^3$ would be written $3 \cdot 3 \cdot 3 \cdot \underline{x \cdot x} + 2 \cdot 2 \cdot 5 \cdot \underline{x \cdot x} \cdot x$. There is no numerical factor that appears in every term, but the variable x appears in every term. The highest power of x that appears in every term is $\underline{x \cdot x}$ or x^2. So, the GCF is x^2.

After the GCF is identified, the polynomial can be factored by *removing* the GCF. The GCF is written outside of a set of parentheses, and a polynomial consisting of remaining factors is written inside the parentheses.

- $30x^2 + 12x - 18 = \underline{2 \cdot 3} \cdot 5 \cdot x \cdot x + 2 \cdot \underline{2 \cdot 3} \cdot x - \underline{2 \cdot 3} \cdot 3 = \underline{2 \cdot 3}(5 \cdot x \cdot x + 2 \cdot x - 3) = 6(5x^2 + 2x - 3)$
- $27x^2 + 20x^3 = 3 \cdot 3 \cdot 3 \cdot \underline{x \cdot x} + 2 \cdot 2 \cdot 5 \cdot \underline{x \cdot x} \cdot x = \underline{x \cdot x}(3 \cdot 3 \cdot 3 + 2 \cdot 2 \cdot 5 \cdot x) = x^2(27 + 20x)$

The outcome of factoring a polynomial by removing the GCF is a product of a monomial and another polynomial. To check the results, do the following:

- Be sure that there is no common factor in all of the terms of the polynomial inside the parentheses. For example, $30x^2 + 12x - 18 = 3(10x^2 + 4x - 6)$, but $3(10x^2 + 4x - 6)$ is not in completely factored form because all the terms of $10x^2 + 4x - 6$ have the common factor two. In other words, three is not the GCF of $30x^2 + 12x - 18$. Its GCF is six.

- Multiply the monomial and the polynomial that is inside the parentheses; the product should be the original polynomial. For example, $3(5x^2 + 2x - 3)$ is not the correct result of factoring $30x^2 + 12x - 18$ because the product of three and $5x^2 + 2x - 3$ is $15x^2 + 6x - 9$, not $30x^2 + 12x - 18$.

1 **Factor: $6x + 12$.** Writing each term as a product of prime factors, the first term is written $2 \cdot 3 \cdot x$, and the second term is written $2 \cdot 2 \cdot 3$. Both have a common factor two and a common factor three. Therefore, the GCF is $2 \cdot 3$ or 6. After removing the two and three from each term, what remains is x and two. Thus, the factored form is $6(x + 2)$.

2 **Factor: $5x^4 - 15x^2 - 10$.** Writing each term as a product of prime factors, the first term is written $5 \cdot x \cdot x \cdot x \cdot x$; the second term is written $3 \cdot 5 \cdot x \cdot x$; and the third term is written $2 \cdot 5$. The only common factor is five, so the GCF is five. Removing five from each term leaves $x \cdot x \cdot x \cdot x$, $3 \cdot x \cdot x$, and 2. The factored form is $5(x^4 - 3x^2 - 2)$.

3 **Factor, if possible: $a^3 - b^2$.** Writing each term as a product of prime factors, the first term is written $a \cdot a \cdot a$, and the second term is written $b \cdot b$. These terms have no common monomial factor.

Additional Examples

1. Factor $8a^2b + 3c^2$, if possible.

The two terms of this polynomial have no common factor other than one. If one is removed, the factored form is $1(8a^2b + 3c^2)$. This factored form is a product, but it is not a product of simpler expressions and serves no purpose. Therefore, we say that there is no common monomial factor.

Note: "No common monomial factor" is generally understood to mean "no common monomial factor other than one."

2. Factor $10x^4 + 2x^3 - 6x$, if possible.

$10x^4 + 2x^3 - 6x =$

$2 \cdot 5 \cdot x \cdot x \cdot x \cdot x + 2 \cdot x \cdot x \cdot x - 2 \cdot 3 \cdot x =$

$2 \cdot x(5 \cdot x \cdot x \cdot x + x \cdot x - 3) =$

$2x(5x^3 + x^2 - 3)$

Section 2

Expand Their Horizons

In Section 2, students will see how to factor by removing the greatest common factor from a polynomial that has two variables. (Neither of the polynomials with two variables that appeared in Section 1 had a common factor.) The method explained in Section 1 for identifying the GCF of a polynomial is not shown in Section 2. Consider the polynomial $6x^3y + 6x^2y^2 - 12x^2y^3$. The factoring method of Section 1 for this polynomial is shown below:

$$\begin{aligned} 6x^3y + 6x^2y^2 - 12x^2y^3 &= \underline{2 \cdot 3} \cdot \underline{x \cdot x} \cdot x \cdot \underline{y} + \\ &\quad \underline{2 \cdot 3} \cdot \underline{x \cdot x} \cdot y \cdot \underline{y} - 2 \cdot \\ &\quad \underline{2 \cdot 3} \cdot \underline{x \cdot x} \cdot y \cdot \underline{y} \cdot y \\ &= \underline{2 \cdot 3} \cdot \underline{x \cdot x} \cdot \underline{y}(x + y \\ &\quad - 2 \cdot y \cdot y) \\ &= 6x^2y(x + y - 2y^2) \end{aligned}$$

Common Error Alert

When the DVD addresses the topic of determining the greatest common factor of a polynomial, the following two bulleted phrases appear:

- Largest numerical factor
- Highest degree in each variable

Students may incorrectly interpret these phrases if they forget that the phrases only apply to a factor that is a *common factor* of all the terms in the polynomial. For example, consider the polynomial $4x^3 + 6x^2$. The largest numerical factor that appears is six, and the highest degree of the variable x that appears is three, which is the degree of x^3. If these factors are combined, the result is $6x^3$, which is NOT the greatest common factor of the polynomial because it is not a *common factor* of the terms $4x^3$ and $6x^2$. Remind students that the bulleted phrases apply only to *common factors* of the terms of a polynomial.

The method of Section 2 for this same polynomial, $6x^3y + 6x^2y^2 - 12x^2y^3$, would be as follows:

- The coefficients 6, 6, and 12 have a greatest common factor of six.
- The highest power of x common to every term is x^2.
- The highest power of y common to every term is y^1 (Remind students that if a variable has no exponent, the exponent is understood to be one).
- Therefore, the greatest common factor is $6x^2y^1$ or $6x^2y$. To factor the polynomial, write the greatest common factor followed by a set of parentheses. Inside the parentheses, write the original polynomial with each term divided by the greatest common factor.
- The fractions inside the parentheses should be simplified:
  $6x^3y + 6x^2y^2 - 12x^2y^3 =$
  $6x^3y\left(\frac{6x^3y}{6x^2y} + \frac{6x^2y^2}{6x^2y} - \frac{12x^2y^3}{6x^2y}\right) =$
  $6x^2y(x + y - 2y^2)$
- The fractions inside the parentheses can be simplified either by writing the numerators and denominators in factored form and canceling or by applying the rules of exponents.
- The numerators and denominators are written in factored form and cancelled:
  $\frac{6x^3y}{6x^2y} = \frac{6xxxy}{6xxy} = \frac{\cancel{6}\cancel{x}\cancel{x}x\cancel{y}}{\cancel{6}\cancel{x}\cancel{x}\cancel{y}} = \frac{x}{1} = x$
  $\frac{6x^2y^2}{6x^2y} = \frac{6xxyy}{6xxy} = \frac{\cancel{6}\cancel{x}\cancel{x}\cancel{y}y}{\cancel{6}\cancel{x}\cancel{x}\cancel{y}} = \frac{y}{1} = y$
  $\frac{12x^2y^3}{6x^2y} = \frac{2 \cdot 6xxyyy}{6xxy} = \frac{2 \cdot \cancel{6}\cancel{x}\cancel{x}\cancel{y}yy}{\cancel{6}\cancel{x}\cancel{x}\cancel{y}} = \frac{2y^2}{1} = 2y^2$
- Or the rules of exponents are applied:
  $\frac{6x^3y}{6x^2y} = \frac{6^1x^3y^1}{6^1x^2y^1} = 6^{1-1} \cdot x^{3-2} \cdot y^{1-1} =$
  $6^0 \cdot x^1 \cdot y^0 = 1 \cdot x \cdot 1 = x$
  $\frac{6x^2y^2}{6x^2y} = \frac{6^1x^2y^2}{6^1x^2y^1} = 6^{1-1} \cdot x^{2-2} \cdot y^{2-1} =$
  $6^0 \cdot x^0 \cdot y^1 = 1 \cdot 1 \cdot y = y$
  $\frac{12x^2y^3}{6x^2y} = \frac{2 \cdot 6^1x^2y^3}{6^1x^2y^1} = 2 \cdot 6^{1-1} \cdot x^{2-2} \cdot y^{3-1} =$
  $2 \cdot 6^0 \cdot x^0 \cdot y^2 = 2 \cdot 1 \cdot 1 \cdot y^2 = 2y^2$

When determining the GCF of a polynomial, it may be worth noting that the ***highest*** power of any variable that appears in ***every*** term is the ***lowest*** power of that variable that appears in ***any*** term. For example, the ***highest*** power of x that appears in ***every*** term of $6x^3y + 6x^2y^2 - 12x^2y^3$ is x^2, and x^2 is the ***lowest*** power of x that appears in ***any*** term.

Common Error Alert

Students may mistakenly write "no solution" or "cannot factor" for a polynomial that has no common factor (other than one). The phrase "no solution" is not appropriate because there is no problem to be "solved." The task in this lesson is to factor a polynomial by removing the greatest common factor, if possible. Factoring a polynomial is a process of rewriting it in another form, not solving anything. The phrase "cannot factor" may or may not be correct, depending on the polynomial. For example, the polynomial $a^3 - b^2$ really cannot be factored, but the polynomial $a^2 - b^2$ can be factored: $a^2 - b^2 = (a - b)(a + b)$. This factoring technique and others are addressed in subsequent lessons. Emphasize to students that they should write "no common monomial factor" or "no GCF" for a polynomial that has no common monomial factor.

4 **Factor: $8x^2y^2 - 12x^4y^3$.** The coefficients 8 and 12 have common factors 2 and 4. Therefore, the greatest common numerical factor is four. Both terms have the variable x, and the greatest power of x that appears in both terms is x^2. Both terms also have the variable y, and the greatest power of y that appears in both terms is y^2. So, the GCF is $4x^2y^2$. To factor the polynomial, write the GCF, followed by parentheses containing the original polynomial with each term divided by the GCF. Then, simplify the fractions inside the parentheses:

$$8x^2y^2 - 12x^4y^3 = 4x^2y^2\left(\frac{8x^2y^2}{4x^2y^2} - \frac{12x^4y^3}{4x^2y^2}\right) = 4x^2y^2(2 - 3x^2y)$$

Look Beyond

Factoring by removing the greatest common factor is only one of several methods of factoring a polynomial. In the remaining lessons of this module, students will learn to:

- Factor polynomials by grouping
- Factor polynomials that are in the form $a^2 - b^2$ (the difference of two squares)
- Factor polynomials of the form $x^2 + bx + c$
- Factor polynomials of the form $ax^2 + bx + c$, in which $a \neq 0$ and $a \neq 1$.
- Factor polynomials by several methods.

Many polynomials require more than one factoring method; for polynomials of this type, removing the greatest common factor is the first method to be used.

Factoring is used throughout mathematics. It is used in geometry to solve problems involving area, in trigonometry to solve equations, and in calculus to find the limits of a function.

Additional Examples

1. Factor: $30q^4r^2s^2 - 15q^3r^2s + 45q^3r^5s^2$.

The coefficients have common factors 3, 5, and 15; 15 will be the numerical part of the GCF. All terms have the variable q, and q^3 will be part of the GCF because q^3 is the highest power of q that appears in every term. Similarly, the GCF will contain r^2 and s. So, the GCF is $15q^3r^2s$.

$30q^4r^2s^2 - 15q^3r^2s + 45q^3r^5s^2 =$

$15q^3r^2s\left(\frac{30q^4r^2s^2}{15q^3r^2s} - \frac{15q^3r^2s}{15q^3r^2s} + \frac{45q^3r^5s^2}{15q^3r^2s}\right) =$

$15q^3r^2s(2qs - 1 + 3r^2s)$

2. Factor: $6y^2z + 21yz^2$.

The only common numerical factor of the coefficients 6 and 21 (other than one) is three. Both terms have the variable y, and the highest power of y that appears in both terms is y or y^1. Both terms also have the variable z, and the highest power of z that appears in both terms is z or z^1. Therefore, the GCF is $3yz$.

$6y^2z + 21yz^2 =$

$3yz\left(\frac{6y^2z}{3yz} + \frac{21yz^2}{3yz}\right) =$

$3yz(2y + 7z)$