11.4 teacher notes

Objectives

- Find the product of two monomials.
- Find the product of a monomial and a binomial.
- Find the product of two binomials using the Distributive Property twice.
- Find the product of two binomials using the FOIL Method.

Vocabulary

Commutative Property of Multiplication (Lesson 2-3)
Associative Property of Multiplication (Lesson 2-3)
Multiplication rule exponents (Lesson 11-1)
Base (Lesson 1-4)
Distributive Property (Lesson 2-3)
Like terms (Lesson 2-4)
FOIL Method

Prerequisites

Applying the Commutative Property

Applying the Associative Property

Using the Laws of Exponents

Applying the Distributive Property of Multiplication over Addition

Combining like terms

Get Started

- Separate the class into pairs or small groups. Assign each group one (or more) of the following tasks:
  Define and give an example for the Commutative Property of Multiplication, the Associative Property of Multiplication, and the Distributive Property of Multiplication over Addition. Then, complete the following and state the rule that applies in each case:
  $x^2 \cdot x^3 = ?$, $x = x^?$.
- After checking results, have a representative of each group share their results with the class.

- Acceptable answers include:
  Commutative Property: The order of factors does not affect the result.
  $2 \cdot 3 = 3 \cdot 2$
  Associative Property: The grouping of factors does not affect the result.
  $2(3 \cdot 4) = (2 \cdot 3)4$
  Distributive Property: Multiply each term inside parentheses by the factor outside the parentheses. $2(3 + 4) = 2 \cdot 3 + 2 \cdot 4$
  $x^2 \cdot x^3 = x^5$. To multiply powers with like bases, add the exponents (*multiplication rule of exponents*).
  $x = x^1$. An unwritten exponent is understood to be one.

Section 1

Expand Their Horizons

In Section 1, students will multiply monomials. To multiply monomials, the Commutative and Associative Properties will be used, as well as the *multiplication rule of exponents*. The following concepts, from Lessons 2-3 and 11-4, should be reviewed:

- The Commutative Property of Multiplication states for all real numbers a and b, $ab = ba$. That is, the Commutative Property allows reordering of factors.
- The Associative Property of Multiplication states for all real numbers a, b, and c, $(a \cdot b)c = a(b \cdot c)$. That is, the Associative Property allows regrouping of factors.
- The *multiplication rule of exponents* states for all real numbers a, b, and c, $(a \neq 0)$ $a^b \cdot a^c = a^{b+c}$. That is, when multiplying powers with like bases, the exponents are added.
- A base with no exponent is understood to have an exponent of one. That is $x = x^1$.

To multiply monomials, use the Commutative and Associative Properties to get numerical factors together and like bases together. All steps should probably be shown for students in the beginning, gradually omitting steps until the work can be done mentally.

Common Error Alert

Students may mistakenly apply an exponent of zero instead of one to a variable for which no exponent is shown. Stress that any number to the first power is itself: $2^1 = 2$, $7^1 = 7$, and $x^1 = x$. It may help to remind students that an exponent indicates the number of times to use a base as a factor. The expression x^4 means $x \cdot x \cdot x \cdot x$. Likewise, the expression x^1 means x.

Consider the example $5x^3y^2z \cdot 2xy^3z$. Using the Commutative Property, reorder the factors to get like factors together: $5 \cdot 2 \cdot x^3 \cdot x \cdot y^2 \cdot y^3 \cdot z \cdot z$. Using the Associative Property, group like factors together: $(5 \cdot 2)(x^3 \cdot x)(y^2 \cdot y^3)(z \cdot z)$. Recall that a variable with no exponent shown is understood to have an exponent of one: $(5 \cdot 2)(x^3 \cdot x^1)(y^2 \cdot y^3)(z^1 \cdot z^1)$. Now multiply; remembering that to multiply variables with like bases, add the exponents: $10(x^{3+1})(y^{2+3})(z^{1+1}) = 10x^4y^5z^2$.

Consider the example $3b^3c \cdot 7a^2$. The only like factors are the constants; multiplying these gives $21b^3ca^2$. Because of the Commutative Property, the order in which the variables are listed in the expression does not matter; $21b^3ca^2$, $21a^2cb^3$, and $21a^2b^3c$ are all correct. However, the generally accepted convention is to list the numerical coefficient first and then, to list variables in alphabetical order, making $21a^2b^3c$ the best way to write the expression.

1 **Simplify: $x^3y \cdot 6xy^2$.** Applying the Commutative Property, the expression becomes $6 \cdot x^3 \cdot x \cdot y \cdot y^2$. Applying the Associative Property to group like factors, the expression becomes $6(x^3 \cdot x)(y \cdot y^2)$. The variables with no exponent are understood to have an exponent of one: $6(x^3 \cdot x^1)(y^1 \cdot y^2)$. Adding the exponents of like bases, according to the *multiplication rule of exponents*, the expression becomes $6x^4y^3$.

Common Error Alert

When multiplying monomials, coefficients are multiplied, and exponents are added. Because both are numbers, students may have difficulty keeping the distinction clear—sometimes incorrectly adding coefficients or multiplying exponents. Encourage students to write an expression without exponents as an intermediate step to help them avoid errors.

$3x^2 \cdot 5x^4 = 3 \cdot x \cdot x \cdot 5 \cdot x \cdot x \cdot x \cdot x = 15x^6$

$3x^2 \cdot 5x^4 \neq 8x^6$

$3x^2 \cdot 5x^4 \neq 15x^8$

Additional Examples

1. Simplify: $(-5q^3r^2s)(-5q^3r^2s)$.

Applying the Commutative Property to rearrange the factors, the expression becomes $(-5)(-5) \cdot q^3 \cdot q^3 \cdot r^2 \cdot r^2 \cdot s \cdot s$. The Associative Property allows like factors to be grouped together, with an exponent of one understood on the s factors with no exponent shown: $(-5)(-5)(q^3 \cdot q^3)(r^2 \cdot r^2)(s^1 \cdot s^1)$. Multiplying negative five and negative five yields 25 for the coefficient. When multiplying variable factors, the exponents are added $q^3 \cdot q^3 = q^{3+3} = q^6$; $r^2 \cdot r^2 = r^{2+2} = r^4$; and $s^1 \cdot s^1 = s^{1+1} = s^2$. The simplified expression is $25q^6r^4s^2$.

2. Simplify: $(5x^3)^2$.

Squaring a term means to multiply the term by itself; thus, $(5x^3)^2$ means $(5x^3)(5x^3)$. Point out that this example is similar to Example 1, in which a term was multiplied by itself. Using the Commutative Property to change the order of the factors, the multiplication becomes $5 \cdot 5 \cdot x^3 \cdot x^3$. Grouping like factors, as allowed under the Associative Property, the multiplication becomes $(5 \cdot 5)(x^3 \cdot x^3)$. The coefficients are multiplied to get 25, and the exponents are added to indicate the product of the variable factors with like bases: $25(x^{3+3})$. The result is $25x^6$. Note, both additional examples could be done using the property $(a^m)^n = a^{mn}$.

Section 2

Expand Their Horizons

In Section 2, students will use the Distributive Property along with the concepts of multiplying monomials as they multiply a binomial by a monomial. Recall the Distributive Property of Multiplication over Addition states each term inside a set of grouping symbols is multiplied by the factor outside the grouping symbols. As students use this property, they will be multiplying monomials, which was covered in Section 1.

Consider the example $2x^3(3x - 7)$. The binomial $3x - 7$ is the difference of the monomials $3x$ and seven and is to be multiplied by the monomial $2x^3$. Using the Distributive Property, $2x^3$ is "distributed" to each monomial inside the parentheses: $2x^3 \cdot 3x - 2x^3 \cdot 7$. Notice that because $3x - 7$ is treated as a difference, the result, $2x^3 \cdot 3x - 2x^3 \cdot 7$, is treated as a difference. Applying the Commutative and Associative Properties as before, the expression becomes $(2 \cdot 3)(x^3 \cdot x^1) - (2 \cdot 7)x^3$. Adding the exponents to indicate multiplication of x's, the result is $6x^4 - 14x^3$. This expression cannot be simplified further because it has no like terms. Like terms must have not only the same variable but also the same exponent on that variable, and these are not like terms.

It may be helpful to treat the binomial $3x - 7$ as a sum of the monomials $3x$ and -7: $2x^3(3x - 7) = 2x^3[3x + (-7)] = 2x^3 \cdot 3x + 2x^3 \cdot (-7)$. This will yield the same result and may prevent sign errors.

Common Error Alert

Students may not follow through to complete a multiplication. For example, they may get $6x^4 - 14$ as the product for $2x^3(3x - 7)$. The error, in a case like this, is a result of carelessness. Encourage students to write out the intermediate step: $2x^3(3x - 7) = (2x^3)(3x) - (2x^3)(7) = 6x^4 - 14x^3$.

2 **Simplify: $2x(x + 2)$.** Each term in the binomial is multiplied by the monomial $2x$: $2x \cdot x + 2x \cdot 2$. Rearranging and grouping like factors, the expression becomes $2(x \cdot x) + (2 \cdot 2)x$. Each x is understood to have an exponent of one, and the result is $2x^2 + 4x$. Since these are not like terms, they cannot be combined, and this is the final answer.

3 **Simplify: $-4ab(8a - 3b^3)$.** Multiplying each term in the binomial by the monomial $-4ab$ yields $-4ab \cdot 8a + 4ab \cdot 3b^3$. Note, the second term in the result has a positive coefficient because it is the product of two factors with negative coefficients. Applying the Commutative and Associative Properties, the expression becomes $(-4 \cdot 8)(a \cdot a)b + (4 \cdot 3)a(b \cdot b^3)$. Multiplying the coefficients and adding the exponents of like bases, the result is $-32a^2b + 12ab^4$. These are not like terms, so this expression is in simplest form.

Additional Examples

1. Simplify: $12x\left(\frac{3}{4}x - \frac{1}{2}\right)$.

Using the Distributive Property to multiply each term of the binomial by the monomial $12x$, the expression can be rewritten as

$12x \cdot \frac{3}{4}x - 12x \cdot \frac{1}{2}$.

$\left(12 \cdot \frac{3}{4}\right)(x \cdot x) - \left(12 \cdot \frac{1}{2}\right)x$

$(3 \cdot 3)(x \cdot x) - (6)x$

$9(x \cdot x) - 6x$

$9x^2 - 6x$

This result is in simplest form since there are no like terms to combine.

2. Simplify: $ab^2(5a - 2) - b(3a^2b - ab)$.

This expression requires the use of the Distributive Property twice. The first use of the Distributive Property will be to multiply each term in the binomial $5a - 2$ by the monomial ab^2. The second use of the Distributive Property will be to multiply each term in the binomial $3a^2b - ab$ by the monomial $-b$.

$ab^2 \cdot 5a - ab^2 \cdot 2 - b \cdot 3a^2b + b \cdot ab$.

Rearranging and grouping appropriately, by the Commutative and Associative Properties, the expression becomes $5(a \cdot a)b^2 - 2 \cdot a \cdot b^2 - 3 \cdot a^2(b \cdot b) + a(b \cdot b)$ or $5a^2b^2 - 2ab^2 - 3a^2b^2 + ab^2$. Combine like terms. The simplified product is $2a^2b^2 - ab^2$.

Section 3

Expand Their Horizons

In Section 3, students will learn to multiply two binomials. Two methods are taught—the Distributive Property (used twice) and the FOIL Method.

When using the Distributive Property, the second binomial is distributed over each term in the first binomial. For $(a + b)(c + d)$, this application would yield $a(c + d) + b(c + d)$. Then, the Distributive Property is applied again to get $ac + ad + bc + bd$. Finally, like terms are combined. If any can be combined, most often, the like terms are the two middle terms.

Consider the example $(2x + 3)(x + 5)$. Applying the Distributive Property the first time yields $2x(x + 5) + 3(x + 5)$. Applying the Distributive Property the second time yields $2x^2 + 10x + 3x + 15$. Next, look for like terms that may be combined. In this example, the terms $10x$ and $3x$ are like terms which can be combined to get $13x$. Thus, the simplified product of these binomials is $2x^2 + 13x + 15$.

Consider the example $(x - 4)(x - 1)$. Applying the Distributive Property to this expression, it becomes $x(x - 1) - 4(x - 1)$. It may be helpful to rewrite the original expression as $\left[x + (-4)\right]\left[x + (-1)\right]$ and then, $x\left[x + (-1)\right] + (-4)\left[x + (-1)\right]$ to help students keep track of the signs. However, it is more common to use the $x(x - 1) - 4(x - 1)$ notation. Distributing x through the first set of parentheses and negative four through the second set of parentheses, the product becomes $x^2 - 1x - 4x + 4$. The middle terms are like terms that can be combined for a simplified result of $x^2 - 5x + 4$.

The lesson presents another method for multiplying two binomials—the FOIL Method. The FOIL Method is not actually a different method but rather a memory aid for applying the Distributive Property to two binomials (the FOIL Method applies only to the multiplication of two binomials). FOIL is an acronym for "First, Outer, Inner, Last." It is a systematic approach to ensure each term of the second binomial is multiplied by each term of the first binomial. Consider $(3x + 5)(2x - 4)$ to illustrate the method. The F in the acronym is for "First." It represents the product of the first term in one binomial and the first term in the other binomial. In the expression $(\mathbf{3x} + 5)(\mathbf{2x} - 4)$, the "First" terms are $3x$ and $2x$, and their product is $6x^2$. The O is for "Outer." Of the four terms in the expression $(\mathbf{3x} + 5)(2x - \mathbf{4})$, the "Outer" terms are $3x$ and -4,and their product is $-12x$. The I is for "Inner." In the expression $(3x + \mathbf{5})(\mathbf{2x} - 4)$, the "Inner" terms are 5 and $2x$, and their product is $10x$. Finally, the L is for "Last." In $(3x + \mathbf{5})(2x - \mathbf{4})$, the "Last" terms are 5 and -4, and their product is -20. So, the sum of the four products is $6x^2 - 12x + 10x - 20$. Notice, this same result would be achieved by using the Distributive Property twice. Note: the order is exaclty the same. Combining the like terms, the simplified product is $6x^2 - 2x - 20$.

When multiplying two binomials, there are two special cases that can arise. One is the product of conjugates. In this case, the binomials are identical except for a sign. For example, $(3x + 5)(3x - 5)$ represents a product of conjugates. When these binomials are multiplied, the initial product is $9x^2 - 15x + 15x - 25$. Notice, the sum of the middle terms is zero, so the result is simply $9x^2 - 25$. The terms $9x^2$ and 25 are both perfect squares, and $9x^2 - 25$ is a difference, so $9x^2 - 25$ is a difference of squares.

The other special case is the Perfect Square Trinomial. When a binomial is squared, or raised to the 2nd power, the middle term does not cancel, but it is predictable. The result is a Perfect Square Trinomial. This special case is illustrated in Problem 2 of Additional Examples.

4 **Simplify: $(2p - 5)(p - 4)$.** Using the FOIL Method, the first step is to multiply the first terms: $2p \cdot p = 2p^2$. Next, the outer terms are multiplied: $2p \cdot (-4) = -8p$. The inner terms, -5 and p, yield a product of $-5p$. The product of the last terms, -5 and -4, is 20. The result is $2p^2 - 8p - 5p + 20$. Combining the like terms in the middle, the simplified result is $2p^2 - 13p + 20$. The Distributive Property, used twice, would produce the same result.

Look Beyond

Lessons in this module examine the process of multiplying given monomials or binomials to get products. In later modules, the process of factoring will be examined. Factoring is the process of determining what monomials or binomials (or other polynomials) can be multiplied to get a given product. Also, second-degree equations, called quadratic equations, will be solved in later modules. One method of solving quadratic equations requires factoring. So, a thorough understanding of multiplying monomials and binomials is necessary for a understanding of factoring, and a thorough understanding of factoring is necessary for learning to solve quadratic equations.

Additionally, the multiplication techniques learned here are used in other mathematics courses such as geometry, Algebra II, and trigonometry. In geometry, multiplying polynomials is sometimes needed in area problems. In Algebra II, third- and higher-degree equations are solved. In trigonometry, multiplying and factoring polynomials is needed to prove and to apply certain important identities.

Additional Examples

1. Use the Distributive Property twice to simplify: $(3x + 2y)(5x - y)$.

Applying the Distributive Property the first time, the expression is $3x(5x - y) + 2y(5x - y)$. Applying the Distributive Property the second time and adding exponents to multiply those factors with like bases, the result is $15x^2 - 3xy + 10yx - 2y^2$. By the Commutative Property of Multiplication, $10yx$ can be written $10xy$, which helps to show that the middle terms are like terms. Adding these like terms gives $15x^2 + 7xy - 2y^2$ as the final result.

2. Use the FOIL Method to simplify: $(x + 6)^2$.

$(x + 6)^2 = (x + 6)(x + 6)$. First: The first terms of the binomials are multiplied to get $x \cdot x = x^2$. Outer: The outer terms are multiplied to get $x \cdot 6$ or $6x$. Inner: The product of the inner terms 6 and x is $6x$. Last: The last terms of the binomials are multiplied to get 36. The product is $x^2 + 6x + 6x + 36$. When like terms are combined, the simplified result becomes $x^2 + 12x + 36$.

Manipulatives

Multiplication with monomials and binomials can be modeled using algebra tiles. The small square tiles represent one's (constants), with the yellow side being positive and the red side being negative. The rectangular tiles represent x's, with the green side being positive and the red side being negative. The large square tiles represent x^2's, with the blue side being positive and the red side being negative.

The multiplication will be shown using gridlines, as with multiplication tables. One of the factors is placed to the left of one gridline, and the other factor is placed above the other gridline. The product will be written below and to the right of the gridlines. To determine the size and shape of each term in the product, use the following rules:

A small square times a small square is a small square $(1 \cdot 1 = 1)$.

A small square times a rectangle is a rectangle $(1 \cdot x = x)$.

A rectangle times a rectangle is a large square $(x \cdot x = x^2)$.

Remind students of sign rules for multiplication: A positive times a positive equals a positive; a negative times a negative equals a positive; and a positive times a negative equals a negative. Red tiles for all sizes and shapes indicate negative values. Thus, the sign rules for multiplication of colored tiles could be stated as follows: A non-red times a non-red equals a non-red; a red times a red equals a non-red; and a non-red times a red equals a red.

After multiplying, a pair of tiles of the same size and shape with different colors in the product is a zero pair. For example, a non-red small square and a red small square is a zero pair because they represent $1 + (-1) = 0$. A non-red rectangle and a red rectangle is a zero pair because they represent $x + (-x) = 0$. A non-red large square and a red large square is a zero pair because they represent $x^2 + (-x^2) = 0$. To simplify the product, zero pairs should be removed, and like tiles should be grouped and counted together.

Consider the multiplication $-2x(x - 3)$ of a monomial and a binomial. The term $-2x$ is represented by two red rectangular tiles (rectangular tiles are x tiles and red indicates a negative value) placed to the left of the gridline.

The binomial $x - 3$ is represented by one green rectangular tile (green indicates a positive value for the x tile) and three red small square tiles (small square tiles are 1-tiles and red indicates a negative value) placed above the gridline. See Figure 1 for this setup. Pair the first red rectangular tile on the left with the green rectangular tile above. The product of these is a large square tile, and this tile is red to indicate the negative product of a positive and a negative. See Figure 2 for this product. The same result occurs when pairing the second red rectangular tile on the left with the same green rectangular tile above. See Figure 3.

Figure 1

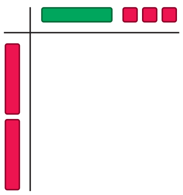


Figure 2

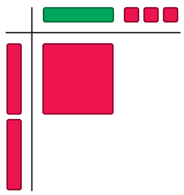
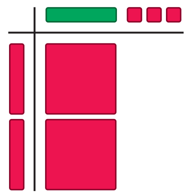


Figure 3



Next, pair the first red rectangular tile on the left with the first red small square tile above. The product of these is a rectangle, and it is green to indicate the positive product of two negatives. See Figure 4 for this product. Repeat this step for the remaining products. See Figure 5 for the final result. There are two large square red tiles (indicating negative x^2 values) and six green rectangular tiles (indicating positive x values), so the product of $-2x(x - 3)$ is $-2x^2 + 6x$.

Figure 4

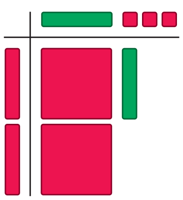
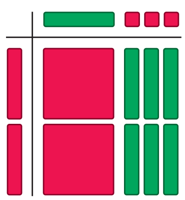


Figure 5



Consider the multiplication $(x + 3)(x - 1)$ of two binomials. The $x + 3$ binomial is represented by one green rectangular tile (for positive x) and three yellow small square tiles (for three positive 1's) placed to the left of the gridline. The $x - 1$ binomial is represented by one green rectangular tile (for positive x) and one red small square tile (for negative one) placed above the gridline. See Figure 6 for this setup. The product of the green rectangular tiles is a blue large square tile (blue indicating positive). See Figure 7. The green rectangular tile on the left, paired with the red small square tile above, yields a red rectangular tile (red indicating negative). See Figure 8 for this product.

Figure 6

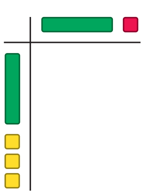


Figure 7

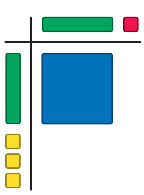
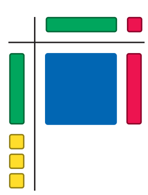


Figure 8



The pairings of each of the three yellow small square tiles on the left with the green rectangular tile above yields three green rectangular tiles. See Figure 9. Finally, the pairings of each of the three yellow small square tiles on the left with the red small square tile above yields three red small square tiles. See Figure 10. Notice, there is a zero pair of rectangular x tiles: one red and one green. These are removed. The tiles that remain are one blue large square tile (for $+1x^2$), two green rectangular tiles (for $+2x$), and three red small square tiles (for –3). The simplified product is $x^2 + 2x - 3$.

Figure 9

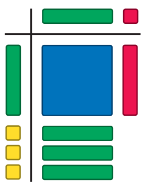


Figure 10

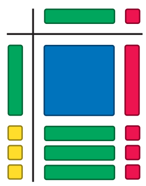


Figure 11